# A SAT-based approach for index calculus on binary elliptic curves

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# Discrete log

## Defining discrete log

Given a finite cyclic group $(G, +)$ and two elements $g$, $h \in G$, find $x \in \mathbb{Z}$ such that

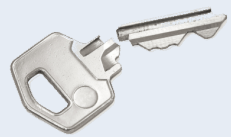
$$h = x \cdot g.$$

## Generic attacks

Pollard rho, Baby-step Giant-step, Kangaroo

## Index calculus attack

Subexponential in $((\mathbb{Z}/p\mathbb{Z})^*, \cdot)$.



# Index calculus on elliptic curves

Let $\mathbb{F}_{2^n}$ be a finite field and $E$ be an elliptic curve defined by

$$E : y^2 + xy = x^3 + ax^2 + b$$

with $a, b \in \mathbb{F}_{2^n}$.

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**Point decomposition phase of the Index calculus algorithm:**

Find $P_1, \ldots, P_{m-1} \in E(\mathbb{F}_{2^n})$, such that

$$P_m = P_1 + \ldots + P_{m-1}$$

# Point Decomposition Problem (PDP)

## Semaev's summation polynomials (2004)

$$S_2(X_1, X_2) = X_1 + X_2,$$
$$S_3(X_1, X_2, X_3) = X_1^2X_2^2 + X_1^2X_3^2 + X_1X_2X_3 + X_2^2X_3^2 + b,$$

For $m \geq 4$

$$S_m(X_1, \ldots, X_m) =$$
$$Res_X(S_{m-k}(X_1, \ldots, X_{m-k-1}, X), S_{k+2}(X_{m-k}, \ldots, X_m, X))$$

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## For $P_1, \ldots, P_m \in E(\mathbb{F}_{2^n})$

$$P_1 + \ldots + P_m = \mathcal{O} \iff S_m(\mathbf{x}_{P_1}, \ldots, \mathbf{x}_{P_m}) = 0$$

# Gaudry and Diem (2008 and 2009)

**Weil descent**

Rewrite the equation $S_m(X_1, \ldots, X_m) = 0$ as a system of $n$ equations over $\mathbb{F}_2$.

Example (trivial case of $m = 2$):

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$$(a_{1,0} + a_{1,1}t + \ldots + a_{1,n-1}t^{n-1}) + (a_{2,0} + a_{2,1}t + \ldots + a_{2,n-1}t^{n-1}) = 0$$

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$$\begin{cases} a_{1,0} + a_{2,0} = 0 \\ a_{1,1} + a_{2,1} = 0 \\ \ldots \\ a_{1,n-1} + a_{2,n-1} = 0 \end{cases}$$

# Gaudry and Diem (2008 and 2009)

## Symmetrization

Rewrite $S_m$ in terms of the elementary symmetric polynomials

$$\begin{aligned} \mathbf{e}_1 &= \sum_{1 \leq i_1 \leq m} X_{i_1}, \\ \mathbf{e}_2 &= \sum_{1 \leq i_1, i_2 \leq m} X_{i_1} X_{i_2}, \\ &\dots \\ \mathbf{e}_m &= \prod_{1 \leq i \leq m} X_i. \end{aligned}$$

# PDP algebraic model

Choice of a factor base : an $l$-dimensional vector subspace $V$ of $\mathbb{F}_{2^n}/\mathbb{F}_2$. When $l \sim \frac{n}{m}$ the system has a reasonable chance to have a solution.

## $X_i$-variables

$$X_1 = a_{1,0} + \ldots + a_{1,l-1}t^{l-1}$$
$$X_2 = a_{2,0} + \ldots + a_{2,l-1}t^{l-1}$$
$$\ldots$$
$$X_m = a_{m,0} + \ldots + a_{m,l-1}t^{l-1}$$

## $\mathbf{e}_i$-variables

$$\mathbf{e}_1 = e_{1,0} + \ldots + e_{1,l-1}t^{l-1}$$
$$\mathbf{e}_2 = e_{2,0} + \ldots + e_{2,2l-2}t^{2l-2}$$
$$\ldots$$
$$\mathbf{e}_m = e_{m,0} + \ldots + e_{m,m(l-1)}t^{m(l-1)}$$

# PDP algebraic model

## Two sets of equations

- Equations defining symmetric polynomials

$$e_{1,0} = a_{1,0} + \ldots + a_{m,0}$$
$$e_{1,1} = a_{1,1} + \ldots + a_{m,1}$$
$$\ldots$$
$$e_{m,m(l-1)} = a_{1,l} \cdot \ldots \cdot a_{m,l}.$$

- Equations derived from the Weil descent

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- Equations derived from the Weil descent

The system is commonly solved using Gröbner basis methods.

# Algebraic model to SAT-reasoning model

Variables in $\mathbb{F}_2$:

$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$.

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$\mathbf{x}_1 + \mathbf{x}_2 \cdot \mathbf{x}_4 + \mathbf{x}_5 \cdot \mathbf{x}_6 + 1 = 0$$
$$\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_4 + \mathbf{x}_5 + 1 = 0$$
$$\mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_2 \cdot \mathbf{x}_4 + 1 = 0$$
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$$\mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_6 + 1 = 0$$

$$(x_1 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge$$
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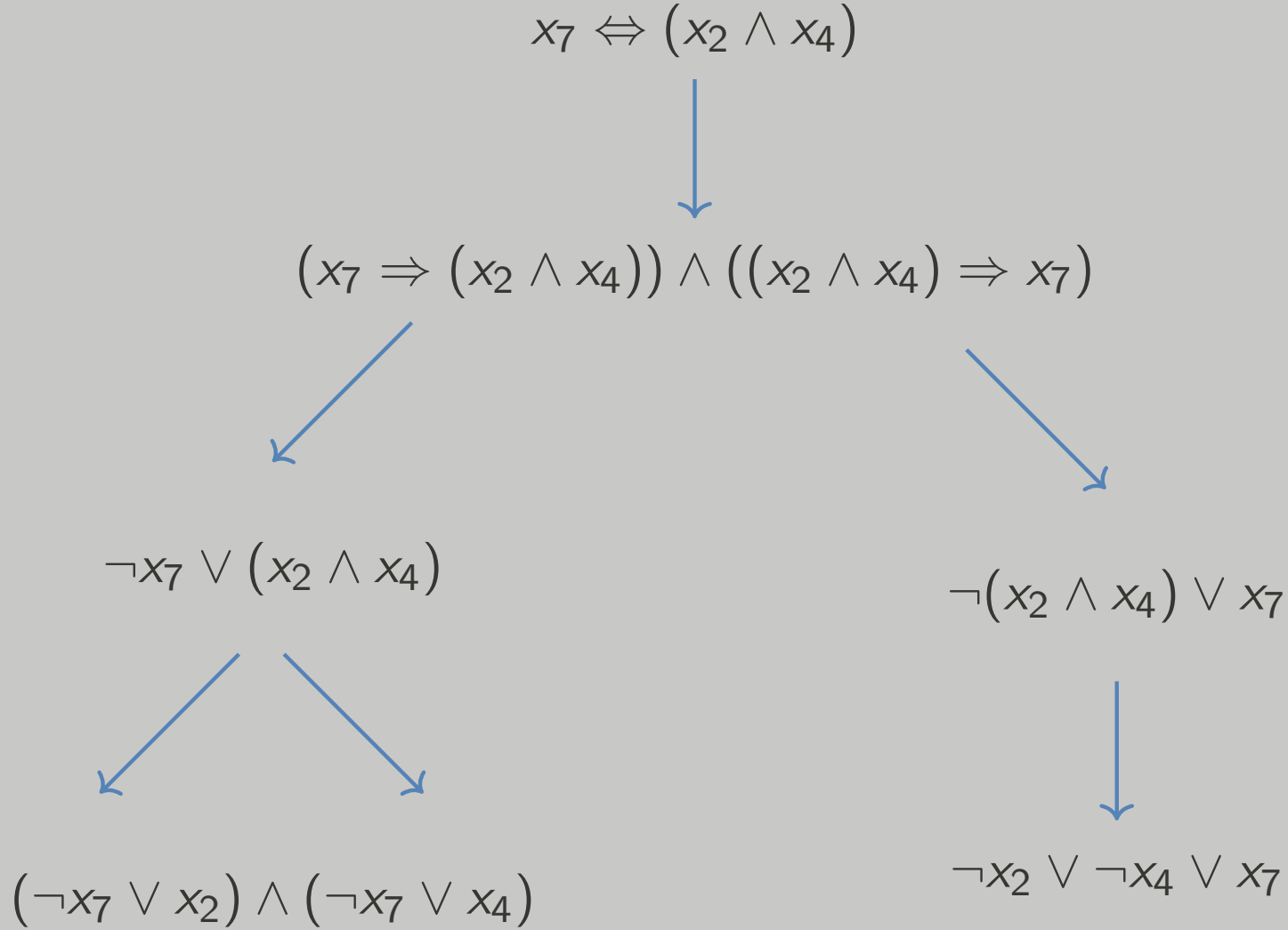
# Algebraic model to SAT-reasoning model

Add new variable $x_7$ to substitute the conjunction $x_2 \wedge x_4$. We have that



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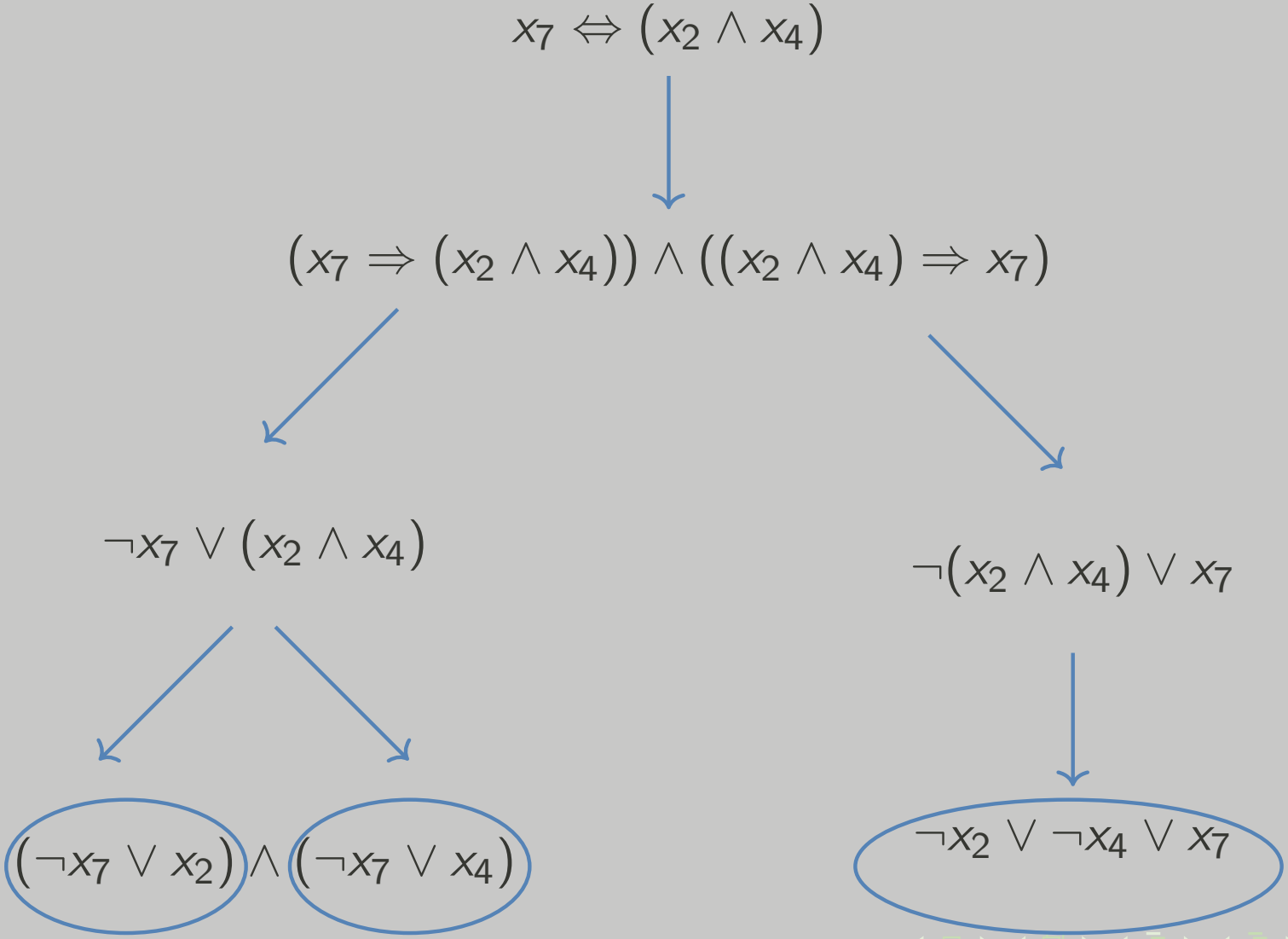

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$$(\neg x_7 \vee x_2) \wedge$$
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$$(\neg x_2 \vee \neg x_4 \vee x_7) \wedge$$
$$(\neg x_8 \vee x_5) \wedge$$
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$$(x_1 \oplus x_7 \oplus x_8) \wedge$$
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# WDSat algorithm

Based on the Davis-Putnam-Logemann-Loveland (DPLL) algorithm.

Recursively building a binary search-tree of height equivalent (at worst) to the number of variables.

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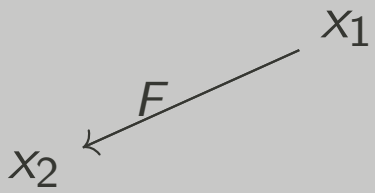

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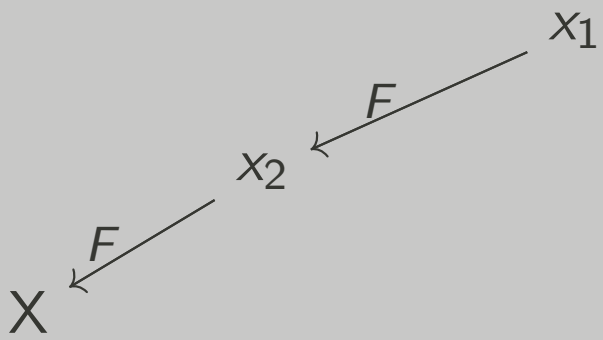

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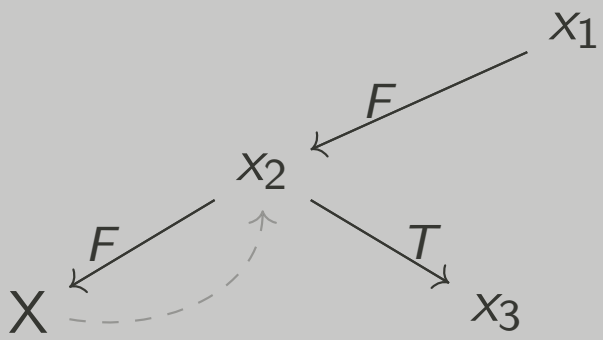

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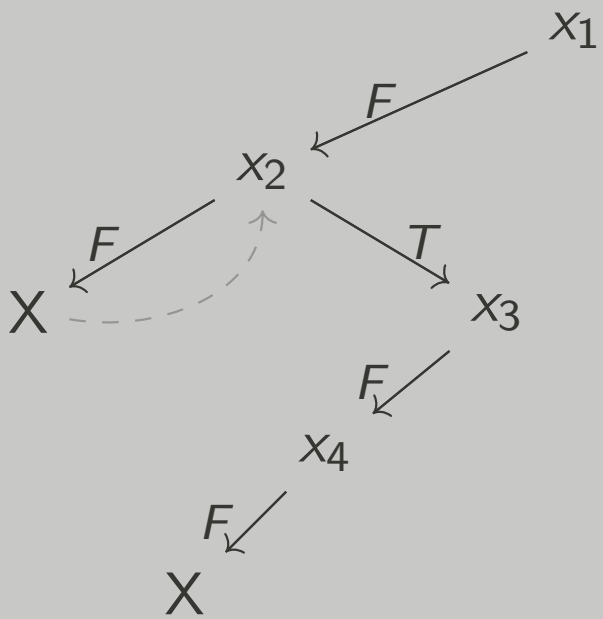

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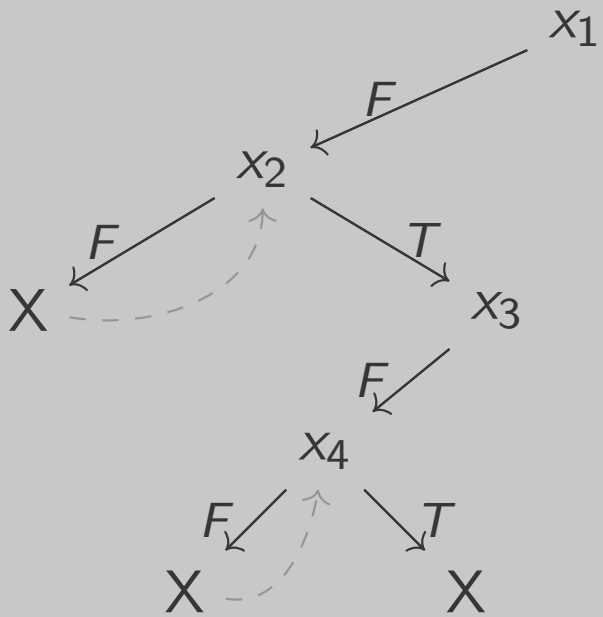

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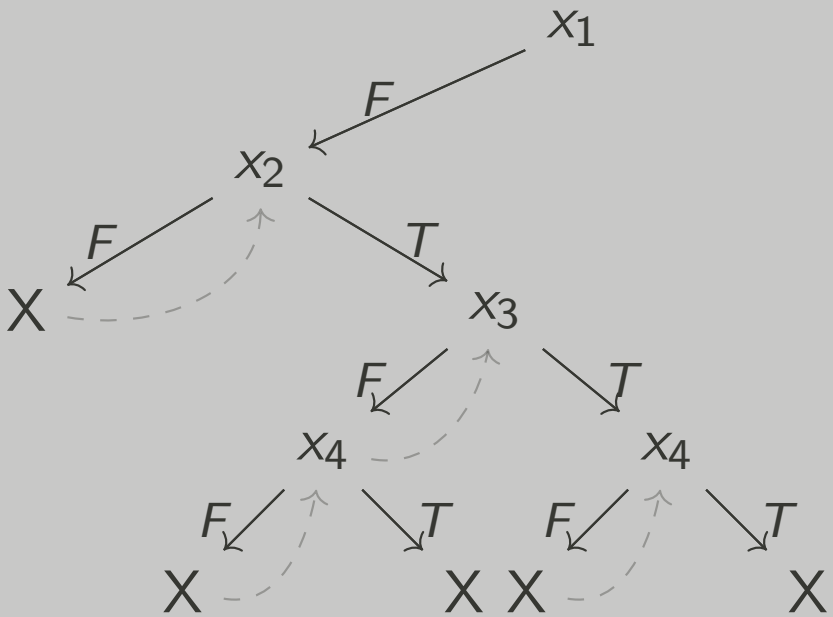

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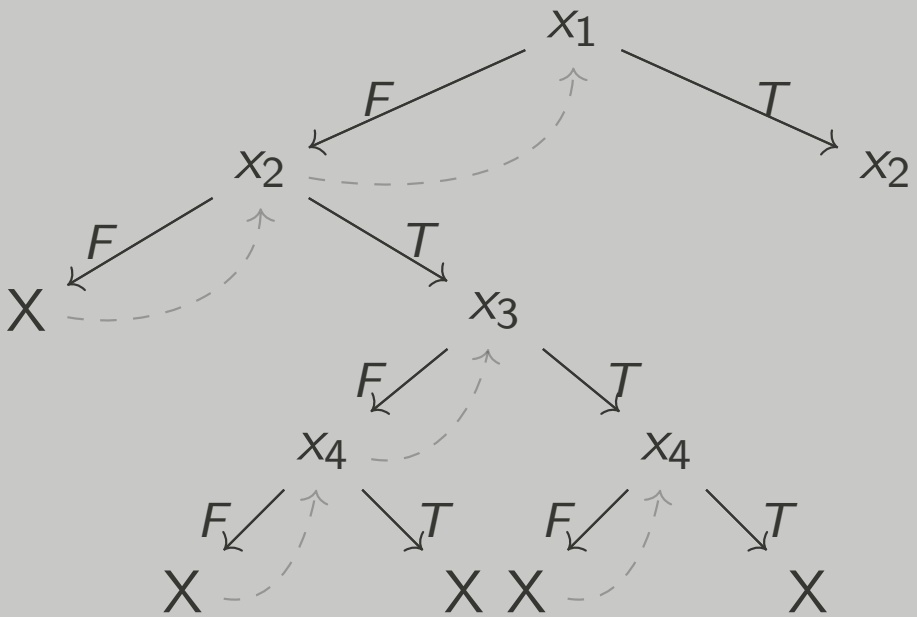

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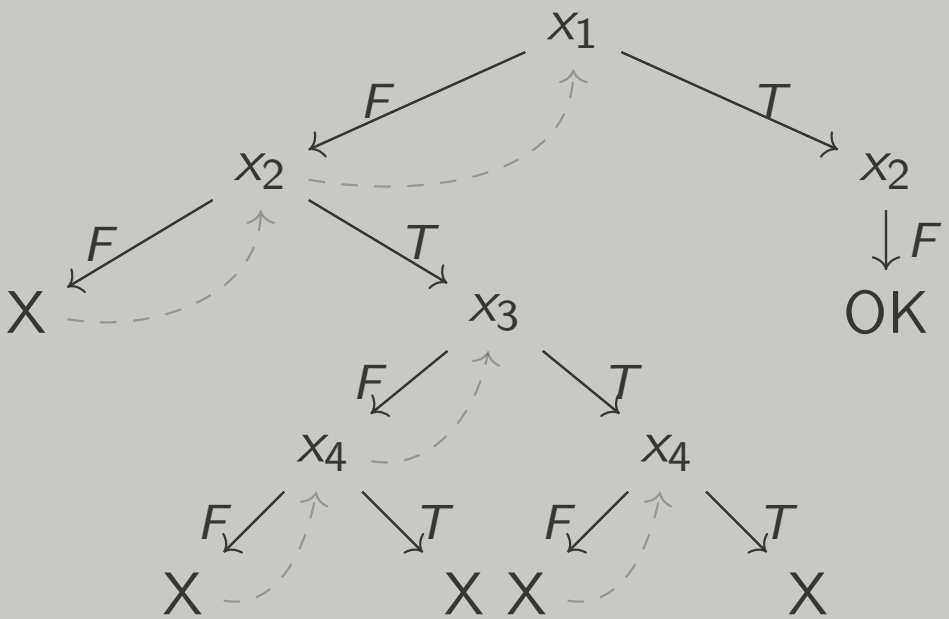

# WDSat - Three reasoning modules

## CNF module

Performs unit propagation on CNF-clauses.

## XORSET module

Performs unit propagation on the parity constraints. When all except one literal in a XOR clause is assigned, we infer the truth value of the last literal according to parity reasoning.

## XORGAUSS module

Performs Gaussian elimination on the XOR system.

# WDSat - breaking symmetry

- Exploiting the symmetry of Semaev's summation polynomials: when $X_1, ..., X_m$ is a solution, all permutations of this set are a solution as well.
- Establish the following constraint $X_1 \leq X_2 \leq \ldots \leq X_m$.
- Implement constraint in the solver using a tree-pruning-like technique.
- Optimizes the complexity by a factor of $m!$.

# Experimental results

| | | | SATisfiable | | | UNSATisfiable | | |
|---|---|---|---|---|---|---|---|---|
| Approach | l | n | Runtime | #Conflicts | Memory | Runtime | #Conflicts | Memory |
| Gröbner basis | 6 | 17 | 207.220 | NA | 3601 | 142.119 | NA | 3291 |
| | | 19 | 215.187 | NA | 3940 | 155.765 | NA | 4091 |
| | 7 | 19 | 3854.708 | NA | 38763 | 2650.696 | NA | 38408 |
| | | 23 | 3128.844 | NA | 35203 | 2286.136 | NA | 35162 |
| WDSat | 6 | 17 | .601 | 49117 | 1.4 | 3.851 | 254686 | 1.4 |
| | | 19 | .470 | 38137 | 1.4 | 3.913 | 255491 | 1.4 |
| | 7 | 19 | 9.643 | 534867 | 16.7 | 44.107 | 2073089 | 16.7 |
| | | 23 | 9.303 | 477632 | 16.7 | 47.347 | 2067168 | 16.7 |
| WDSat breaking-sym | 6 | 17 | .220 | 17792 | 1.4 | .605 | 43875 | 1.4 |
| | | 19 | .243 | 19166 | 1.4 | .639 | 44034 | 1.4 |
| | 7 | 19 | 2.205 | 130062 | 1.4 | 6.859 | 351353 | 1.4 |
| | | 23 | 3.555 | 189940 | 1.4 | 7.478 | 350257 | 1.4 |

Table: Comparing the WDSat approach with the Gröbner basis approach for solving the PDP. Running times are in seconds and memory is in MB.

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| l | n | Runtime | #Conflicts | Memory | Runtime | #Conflicts | Memory |
| 8 | 23 | 29.584 | 1145966 | 17.0 | 81.767 | 2800335 | 17.0 |
| 9 | 37 | 447 | 10557129 | 17.1 | 1048 | 22396994 | 17.1 |
| | 47 | 609 | 12675174 | 17.2 | 1167 | 22381494 | 17.2 |
| | 59 | 611 | 11297325 | 17.3 | 1327 | 22390211 | 17.3 |
| | 67 | 677 | 11608420 | 17.4 | 1430 | 22388053 | 17.4 |
| 10 | 47 | 5847 | 95131900 | 17.3 | 11963 | 179019409 | 17.3 |
| | 59 | 6849 | 97254458 | 17.4 | 13649 | 179067171 | 17.4 |
| | 67 | 6530 | 88292215 | 17.4 | 14555 | 179052277 | 17.4 |
| | 79 | 7221 | 86174432 | 17.5 | 16294 | 179043408 | 17.5 |
| 11 | 59 | 64162 | 727241718 | 19.2 | 135801 | 1432191354 | 19.2 |
| | 67 | 70075 | 741222864 | 19.3 | 145357 | 1432183842 | 19.3 |
| | 79 | 61370 | 599263451 | 19.4 | 161388 | 1432120827 | 19.4 |
| | 89 | 85834 | 736610196 | 19.5 | 175718 | 1432099666 | 19.5 |

Table: Experimental results using the WDSAT solver with breaking symmetry. Running times are in seconds and memory is in MB.

# Conclusion

- When solving the PDP for prime degree extension fields $\mathbb{F}_2$, Gröbner basis methods can be replaced with a SAT-based approach.
- The dedicated SAT-solver, WDSAT, yields significantly faster running times.
- The memory is no longer a constraint for the PDP.
- Preprint at `https://eprint.iacr.org/2019/313`