Isogeny-based cryptography

Monika Trimoska

Selected Areas in Cryptology - Part 1

Spring, 2024

TU/e

Elliptic curves

What is an elliptic curve?

An elliptic curve is an algebraic curve that admits an affine equation of the form

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

(general form of a Weierstrass curve)

with $a_i \in k$, where k is the field where the point is defined.

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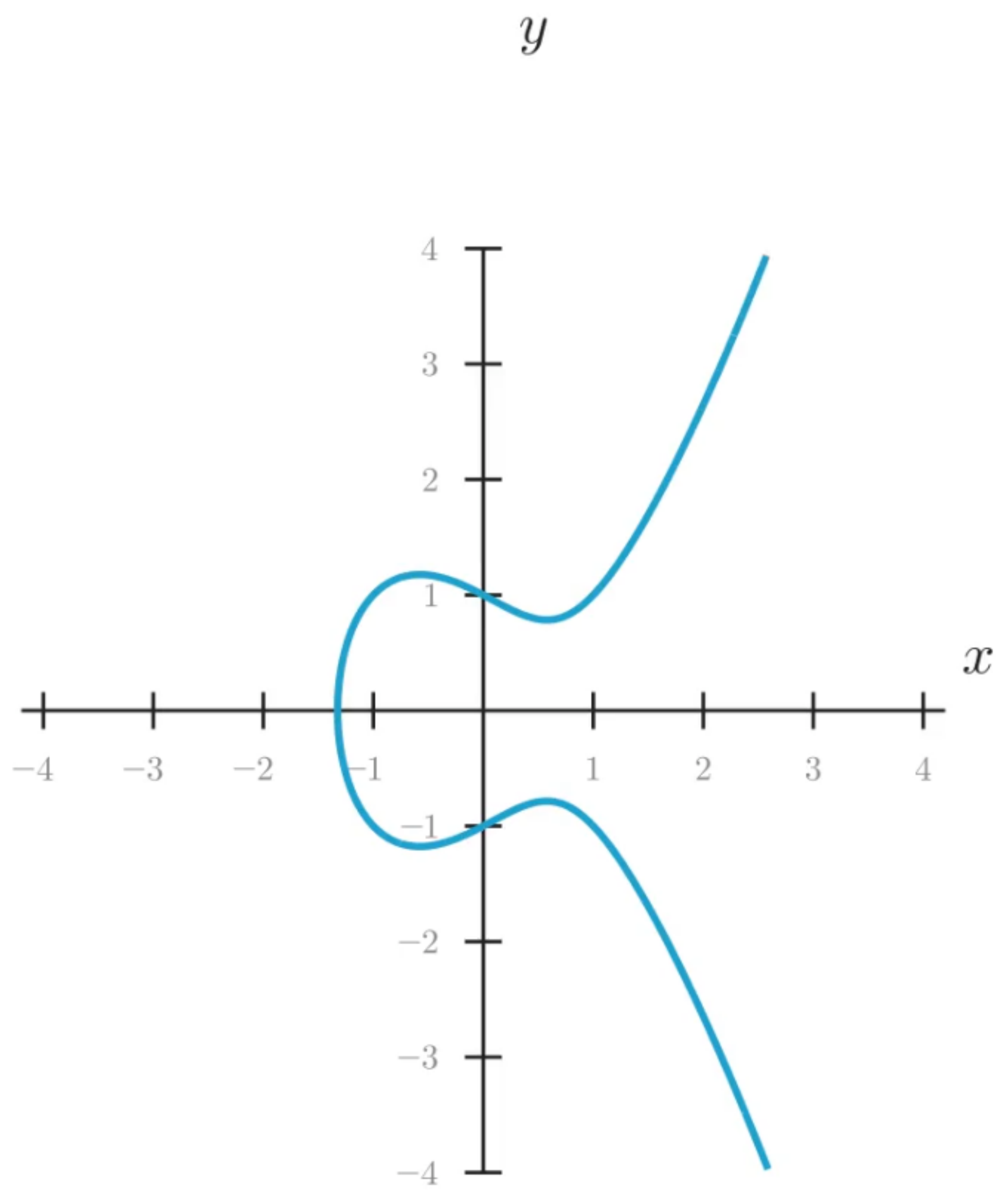

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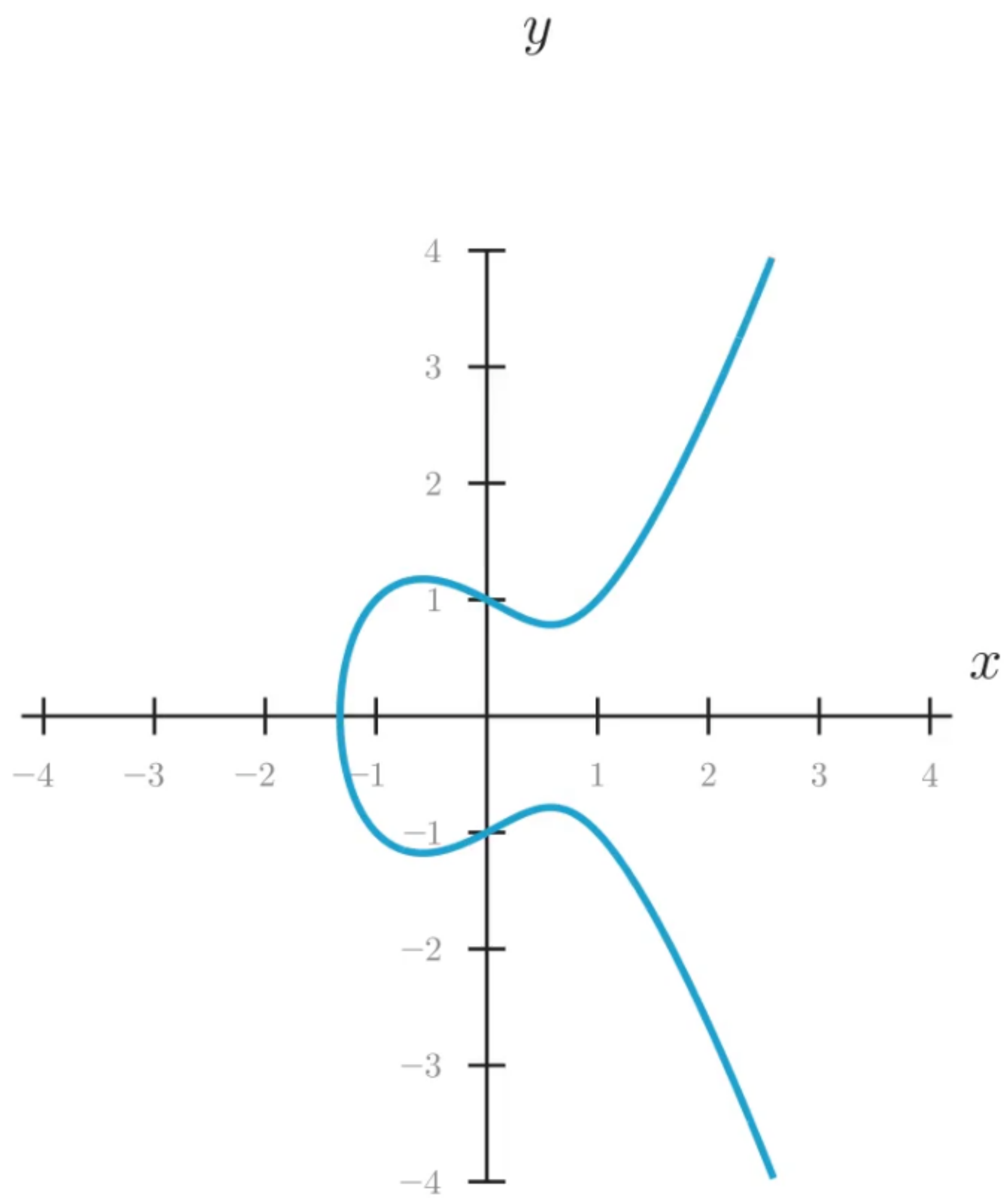

→ A point on E means that the point (x, y) satisfies the curve equation.

Elliptic curves over $\mathbb{F}_q$

In cryptography, we use elliptic curves over finite fields $\mathbb{F}_q$, $q = p^k$ (but we draw the figures over $\mathbb{R}$ because it's nicer).

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Hasse bound

$$\#E(\mathbb{F}_q) \in [q + 1 - 2\sqrt{(q)}, q + 1 + 2\sqrt{(q)}]$$

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  $E[n] = \{P \in E(\overline{\mathbb{F}_p}) \mid nP = \infty\}$ (the n-torsion group)

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Supersingular curves and cyclic groups

- $E(\mathbb{F}_p) \cong \mathbb{Z}/(p+1)$
- $E(\mathbb{F}_{p^2}) \cong \mathbb{Z}/(p+1) \times \mathbb{Z}/(p+1)$

More on elliptic curves

- Short Weierstrass form

$$y^2 = x^3 + c_4 x + c_6$$

⟶ The curve is non-singular if the discriminant $\Delta = 4c_4^3 + 27c_6^2$ is nonzero.

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- The set of points on E with the addition law form a group.

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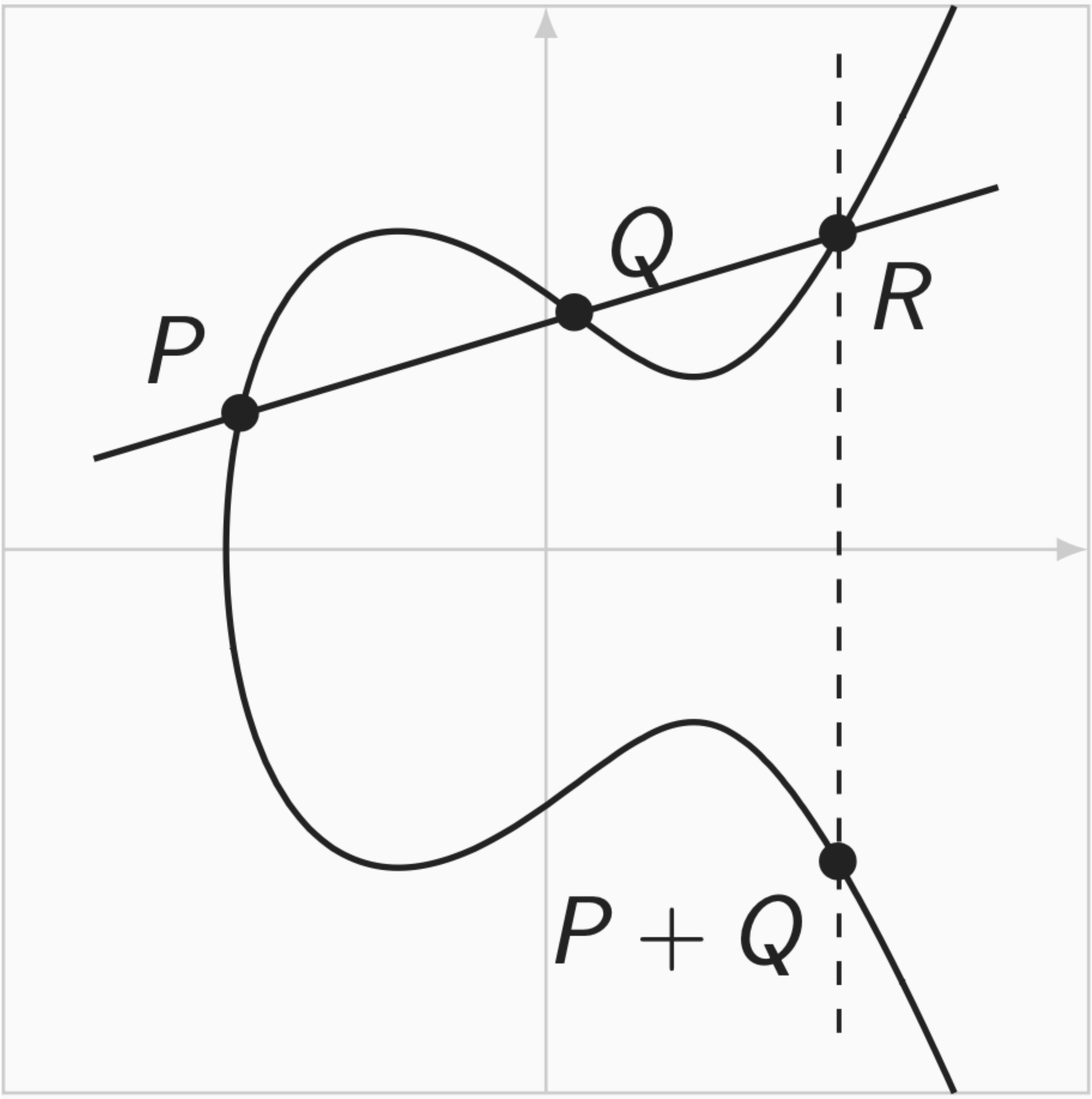
- The set of points on E with the addition law form a group.

- The group law is constructed geometrically.

The geometry of elliptic curves

Adding points on an elliptic curve



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Addition $P + Q$

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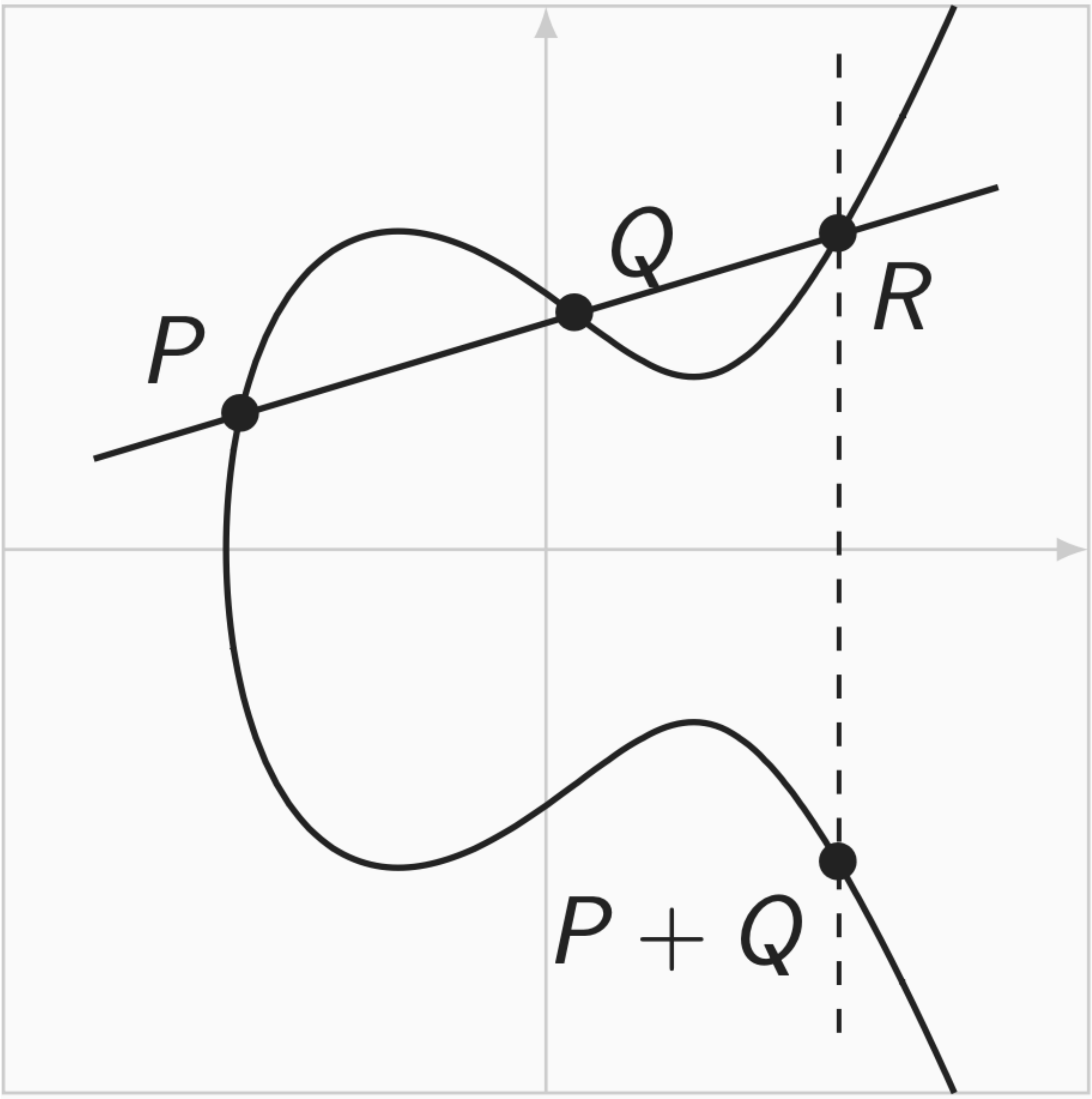

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- Draw a line through P and Q.
  ↪ The line intersects the curve E at a third point R.

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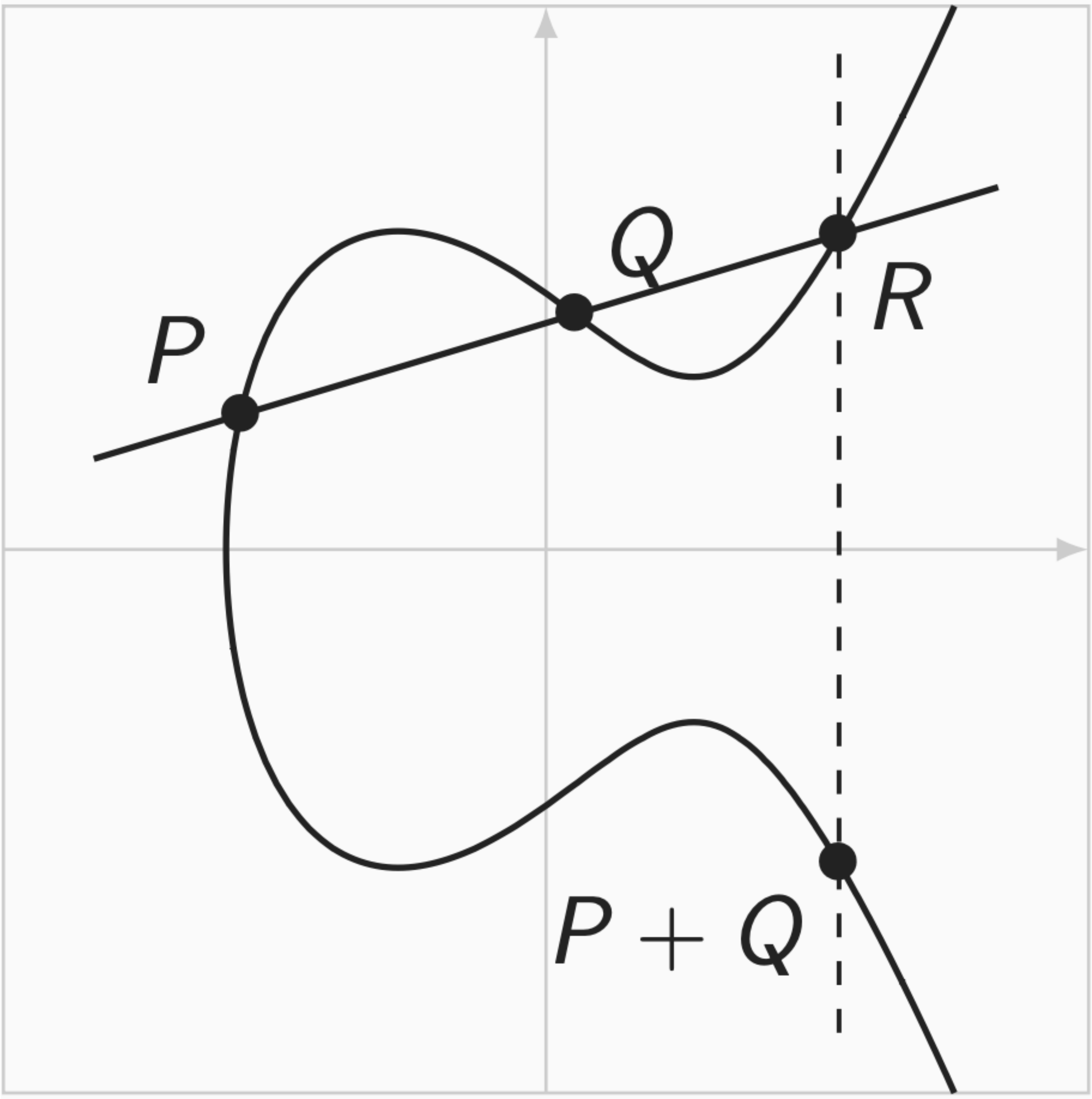

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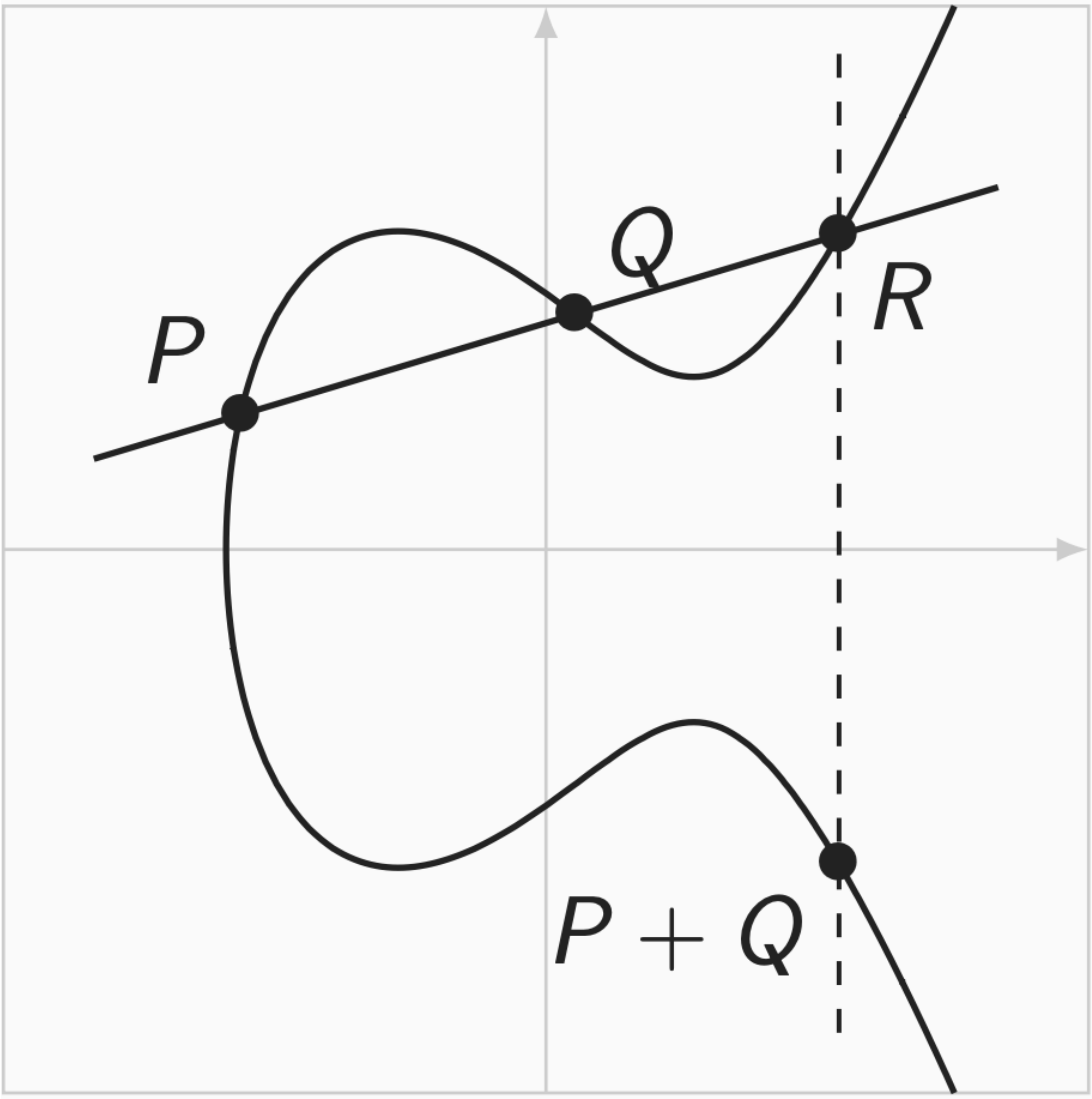

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- Draw a line through P and Q.
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- We define that point to be the sum of P and Q.

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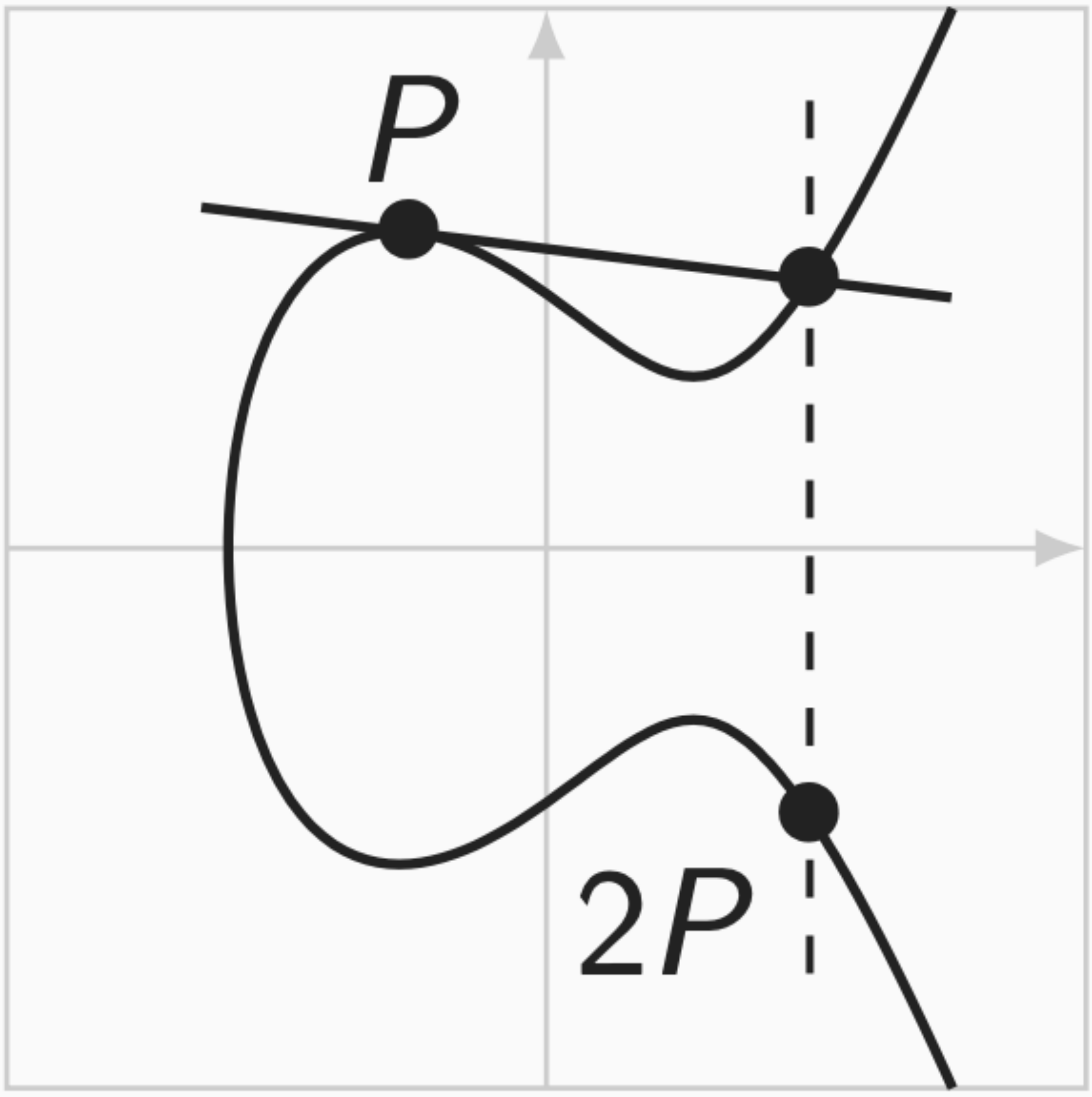

Doubling $P + P$

- Modify the first step: draw the tangent line to E at P.

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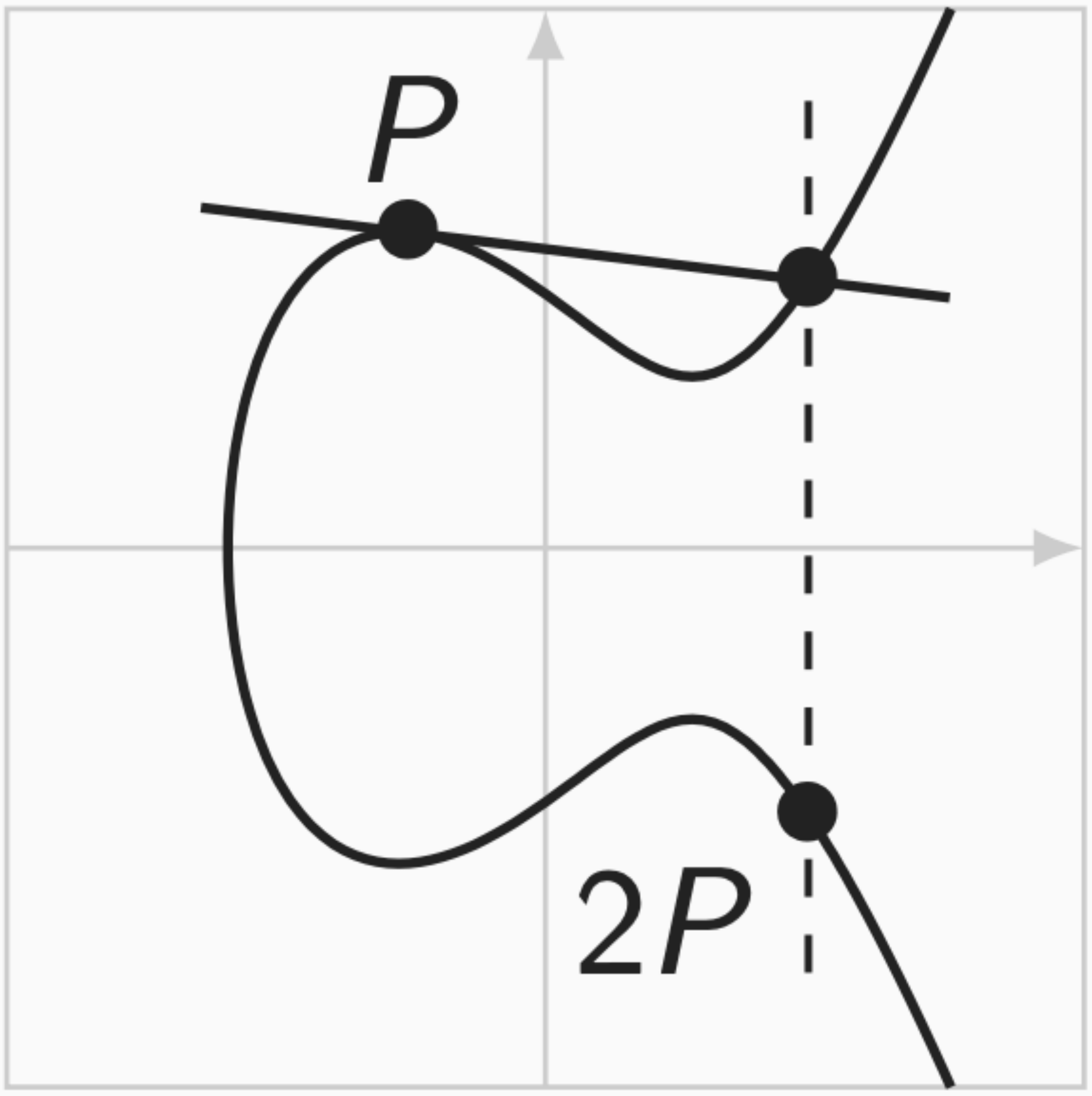

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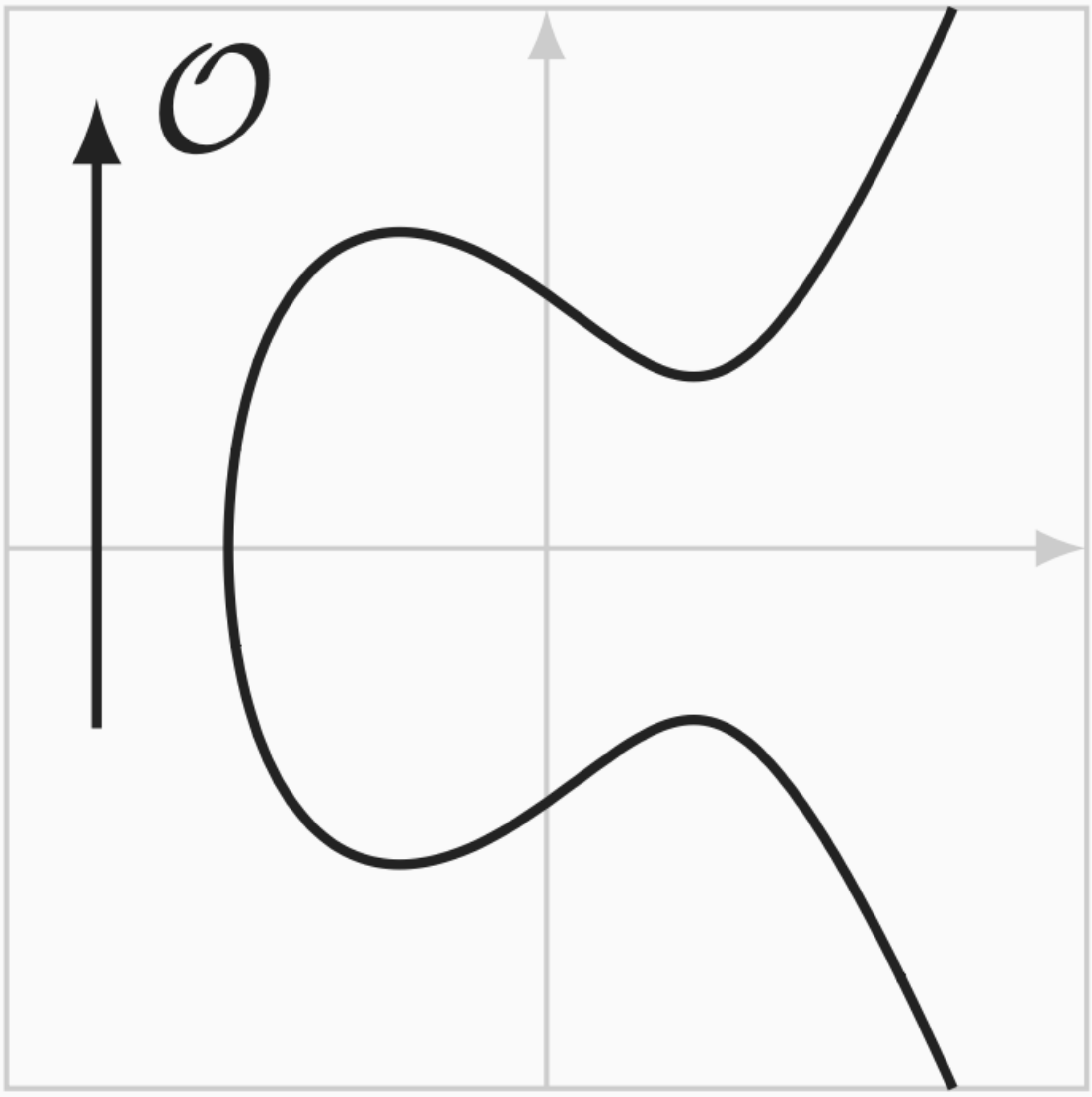

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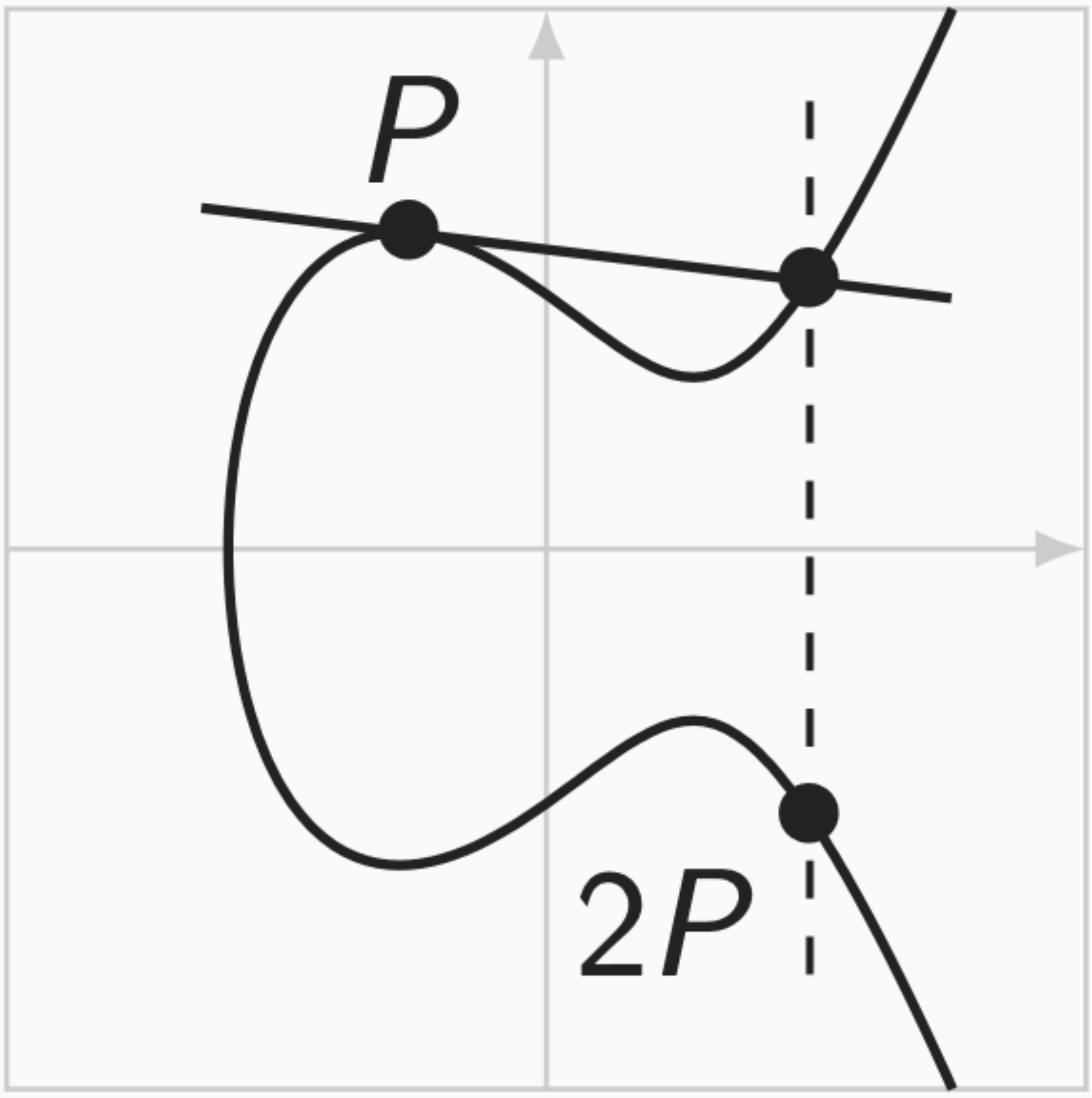

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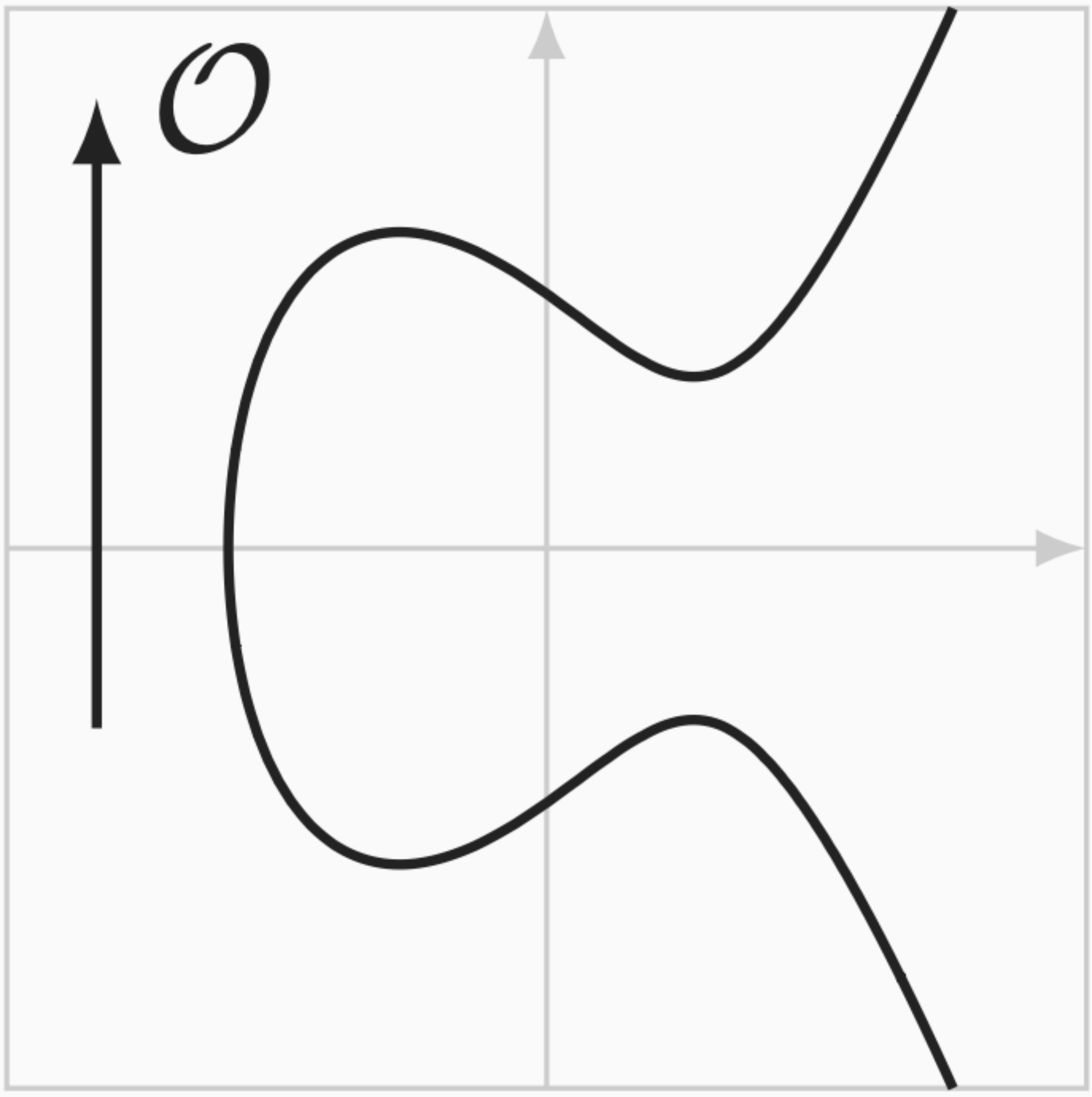

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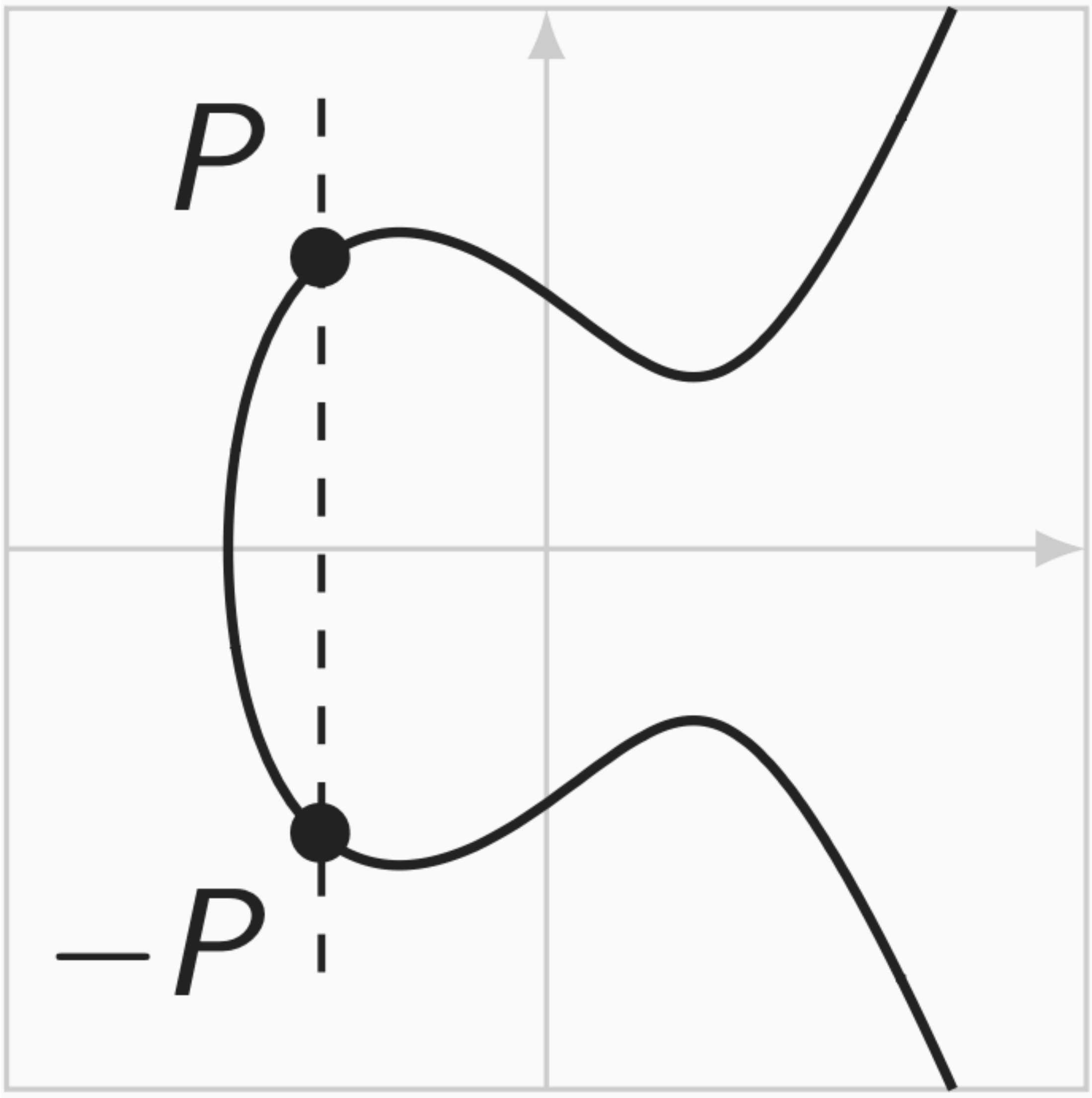

Inverse element $-P$

- Modify the first step: draw the tangent line to E at P.

The algebra of elliptic curves

The addition law on E has the following properties:

- $P + \mathcal{O} = P$, for all $P \in E$
- Let $P \in E$. There is a point on E, denoted by $-P$, satisfying $P + (-P) = \mathcal{O}$.
- $P + (Q + R) = (P + Q) + R$, for all $P, Q, R \in E$
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The arithmetic of elliptic curves

We can write down explicitly the formulas for the addition law on E.

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$,
then $P_1 + P_2 = (x_3, y_3) = (\lambda^2 - x_1 - x_2, \lambda(x_3 - x_1) + y_1)$, where

$$\lambda = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1}, & \text{when } P_1 \neq P_2 \\[2ex] \dfrac{3x_1^2 + a}{2y_1}, & \text{when } P_1 = P_2\,. \end{cases}$$

Elliptic curves in SageMath

Elliptic curves and the group law

```
p=next_prime(2^8)
K=GF(p)
E=EllipticCurve(K, [0, 0, 0, -1, 1])
print(E)
print("Number of points on E:", E.order())
print("E is supersingular: ", E.order()==p+1)
P=E.random_point()
x=K.random_element()
Q=x*P
print("P: ", P, ", Q: ", Q)
print(P+Q == (x+1)*P)
```

0.0s

```
Elliptic Curve defined by y^2 = x^3 + 256*x + 1 over Finite Field of size 257
Number of points on E: 251
E is supersingular:  False
P:  (3 : 5 : 1) , Q:  (48 : 158 : 1)
True
```

Building crypto from elliptic curves (not PQ)

Elliptic curve discrete logarithm problem

The ECDLP problem

Input: Two points $P, Q \in E(\mathbb{F}_q)$.
Question: Find an integer x such that $xP = Q$.

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- First write $m = m_0 + m_1 \cdot 2 + m_2 \cdot 2^2 + \ldots + m_r \cdot 2^r$.
- Then mP can be computed as $mP = m_0 P + m_1 \cdot 2P + m_2 \cdot 2^2 P + \ldots + m_r \cdot 2^r P$.
- Requires $r$doublings (and sums).

Diffie-Hellman key exchange

Alice

Bob

Secret a

Secret b

$A = aP$

$B = bP$

B

A

$K_a = aB$

$K_b = bA$

$$K_a = abP = baP = K_b$$

Isogenies

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j-invariant

- An invariant under isomorphisms.
- For a curve in short Weierstrass form $y^2 = x^3 + c_4 x + c_6$, we have

$$j = 1728 \cdot 4c_4^3/(4c_4^3 + 27c_6^2)$$

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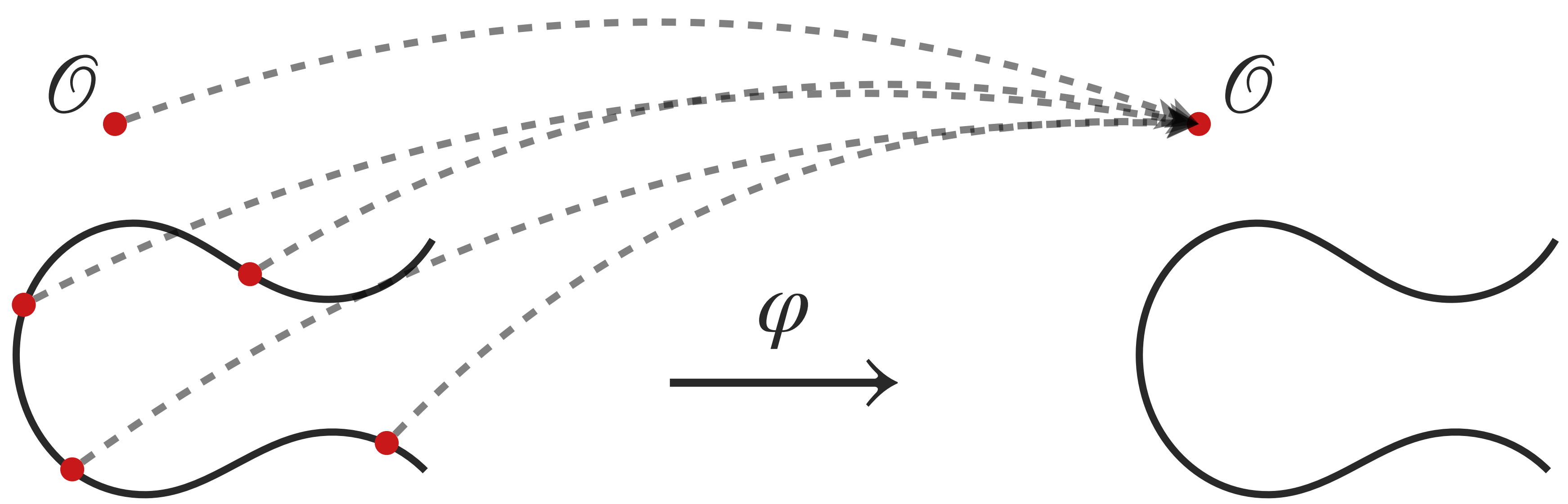

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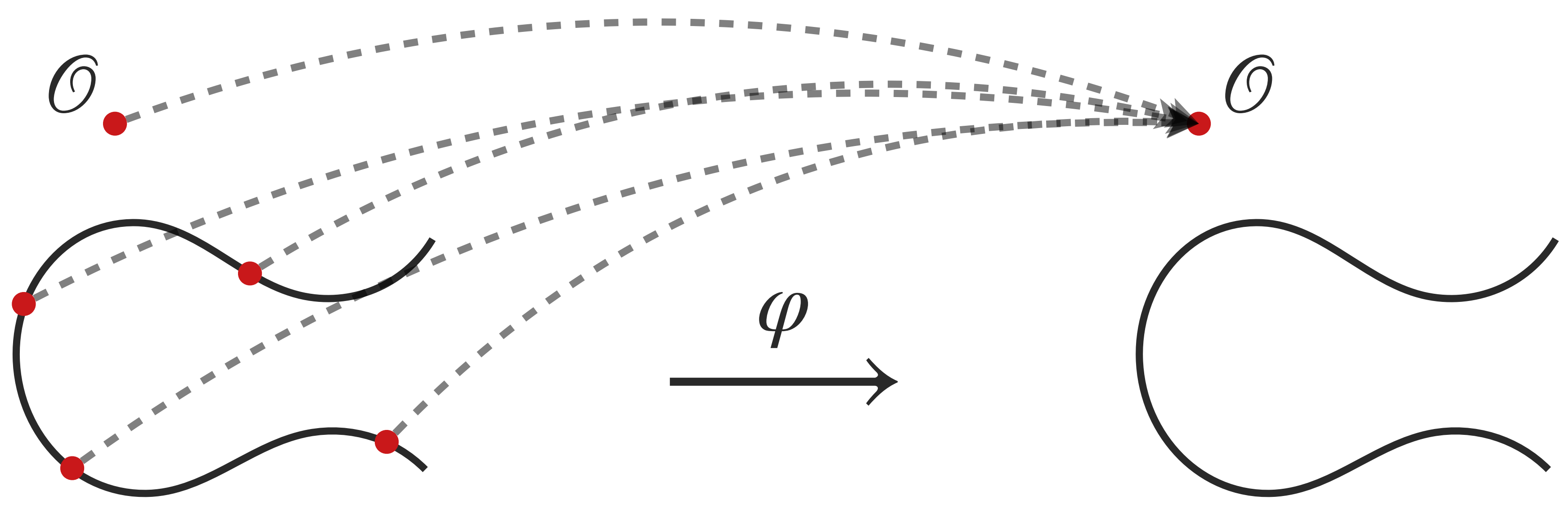
→ An isogeny is uniquely defined by its kernel: $\{P \in E \mid \varphi(P) = \mathcal{O}_{E'}\}$.

→ The degree of a (separable) isogeny is the size of its kernel.



Isogenies

Example.

$$(x, y) \mapsto \left(\frac{x^3 - 4x^2 + 30x - 12}{(x-2)^2}, \frac{x^3 - 6x^2 - 14x + 35}{(x-2)^3} \cdot y\right)$$

defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \to \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2,9), (2, -9), \mathcal{O}\}$.

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ℓ-isogeny:

- $x \to \dfrac{f(x)}{g(x)}$, with $\deg(f) = \ell, \deg(g) = \ell - 1$
- $y \to \ldots$

Computing isogenies

*We consider only supersingular curves from now on.

→ **Goal**: Compute an ℓ-isogeny from E.

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Obtain a kernel of an ℓ-isogeny

↳ Compute an ℓ-isogeny

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Obtain a subgroup of order ℓ

↳ Obtain a kernel of an ℓ-isogeny

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Find a point P on E of order ℓ

↳ Take the cyclic group generated by P

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↳ Compute an ℓ-isogeny

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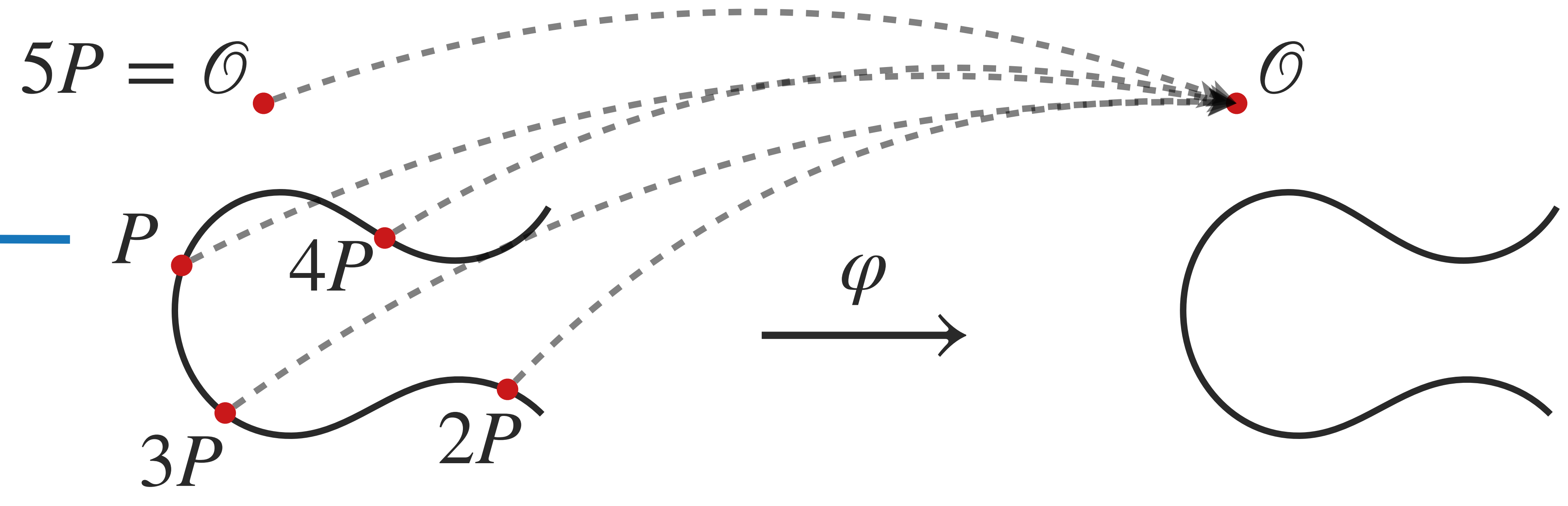
↳ Compute an ℓ-isogeny

point of order 5 ←



Vélu's formulas

→ For any finite subgroup G of E, there exists a unique (up to isomorphism) separable isogeny $\varphi_G : E \to E'$ with kernel G.

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- Formulas for computing E' and evaluating φ_G at a point.
- Complexity: $\Theta(\#G)$ → only suitable for small degrees.

Let P have prime order ℓ on E_A.
For $1 \leq i < \ell$ let x_i be the x-coordinate of iP.
Let

$$\tau = \prod_{i=1}^{\ell-1} x_i, \quad \sigma = \sum_{i=1}^{\ell-1} \left(x_i - \frac{1}{x_i} \right), \quad f(x) = x \prod_{i=1}^{\ell-1} \frac{x x_i - 1}{x - x_i}.$$

Then the ℓ-isogeny with kernel $\langle P \rangle$ is given by

$$\varphi_\ell : E_A \to E_B, (x, y) \mapsto (f(x), c_0 y f'(x))$$

where $B = \tau(A - 3\sigma)$, and $c_0^2 = \tau$.

Composing isogenies

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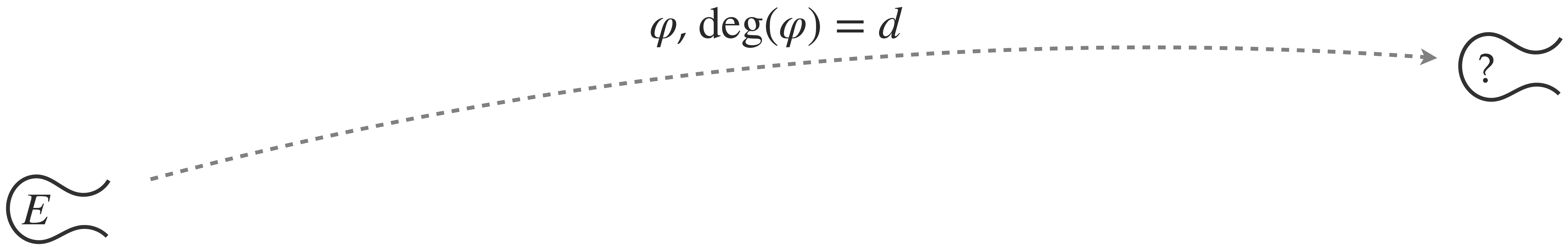

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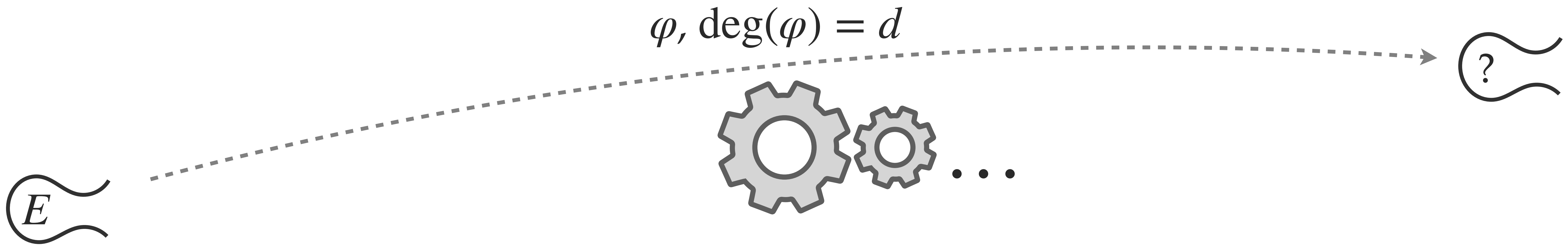
Goal: Compute a d-isogeny from E.



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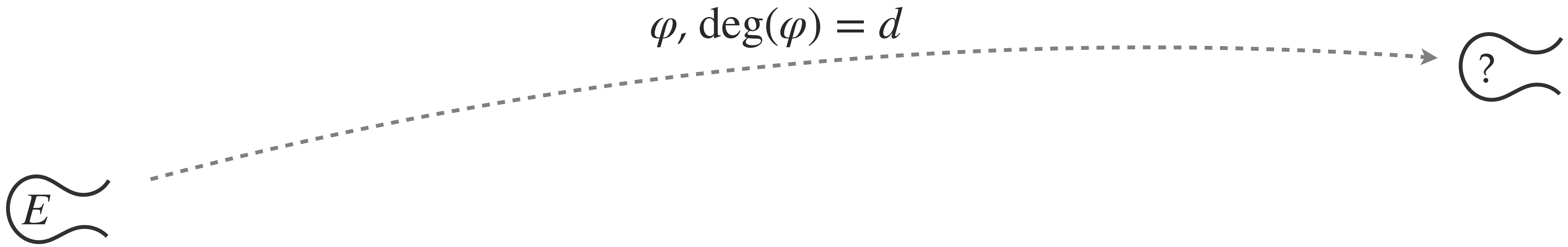

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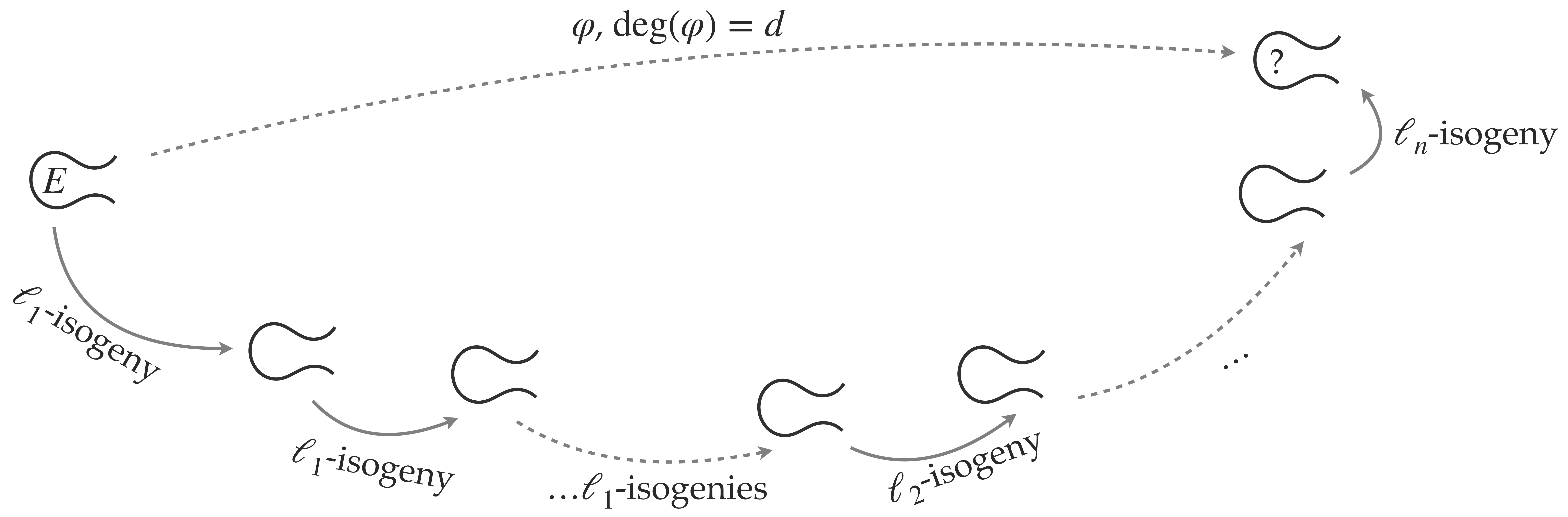

Isogenies in SageMath

Computing isogenies

```
p=139
A=0
E=EllipticCurve(GF(p), [0, A, 0, 1, 0])
assert E.order()==p+1 #check that it is a supersingular curve
print("We can compute isogenies of the following degrees:", factor((p+1)/4))
P=E.random_point()
while P.order().is_prime() == False:
    P=E.random_point()
print("We will compute an isogeny of degree", P.order())
print(E.montgomery_model()) #needs Sage 10.3
phi=E.isogeny(P)
print(phi)
E2=phi.codomain()
print(E2.montgomery_model()) #needs Sage 10.3
```

✓ 0.0s

```
We can compute isogenies of the following degrees: 5 * 7
We will compute an isogeny of degree 5
Elliptic Curve defined by y^2 = x^3 + x over Finite Field of size 139
Isogeny of degree 5 from Elliptic Curve defined by y^2 = x^3 + x over Finite Field of size 139 to Elliptic Curve defined by y^2 = x^3 + 72*x + 30 over Finite Field of size 139
Elliptic Curve defined by y^2 = x^3 + 126*x^2 + x over Finite Field of size 139
```

Endomorphism rings

Dual isogeny

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- The set of endomorphisms forms a ring $\mathrm{End}(E)$ under $+$ and $\circ$.

Hard problems and reductions

The isogeny path problem

Input: Two supersingular curves E and E'.
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The decisional problem is easy (when point counting is easy)

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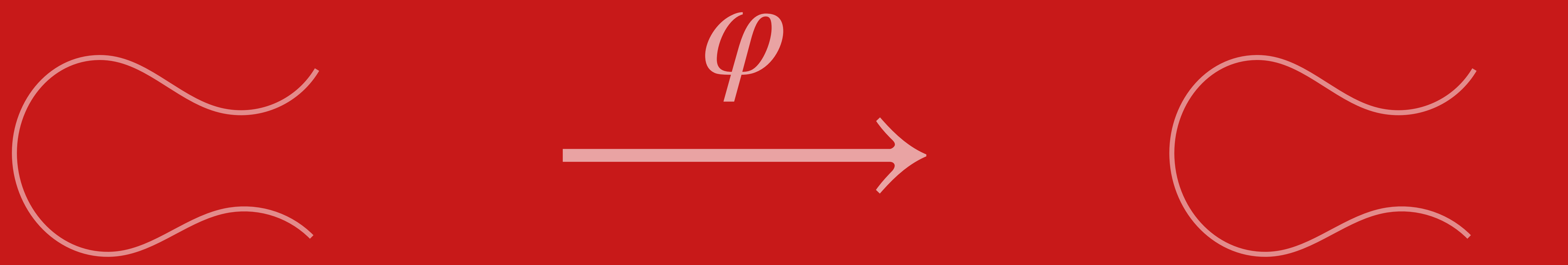
Cryptanalysis

(ECDLP and isogeny path finding)



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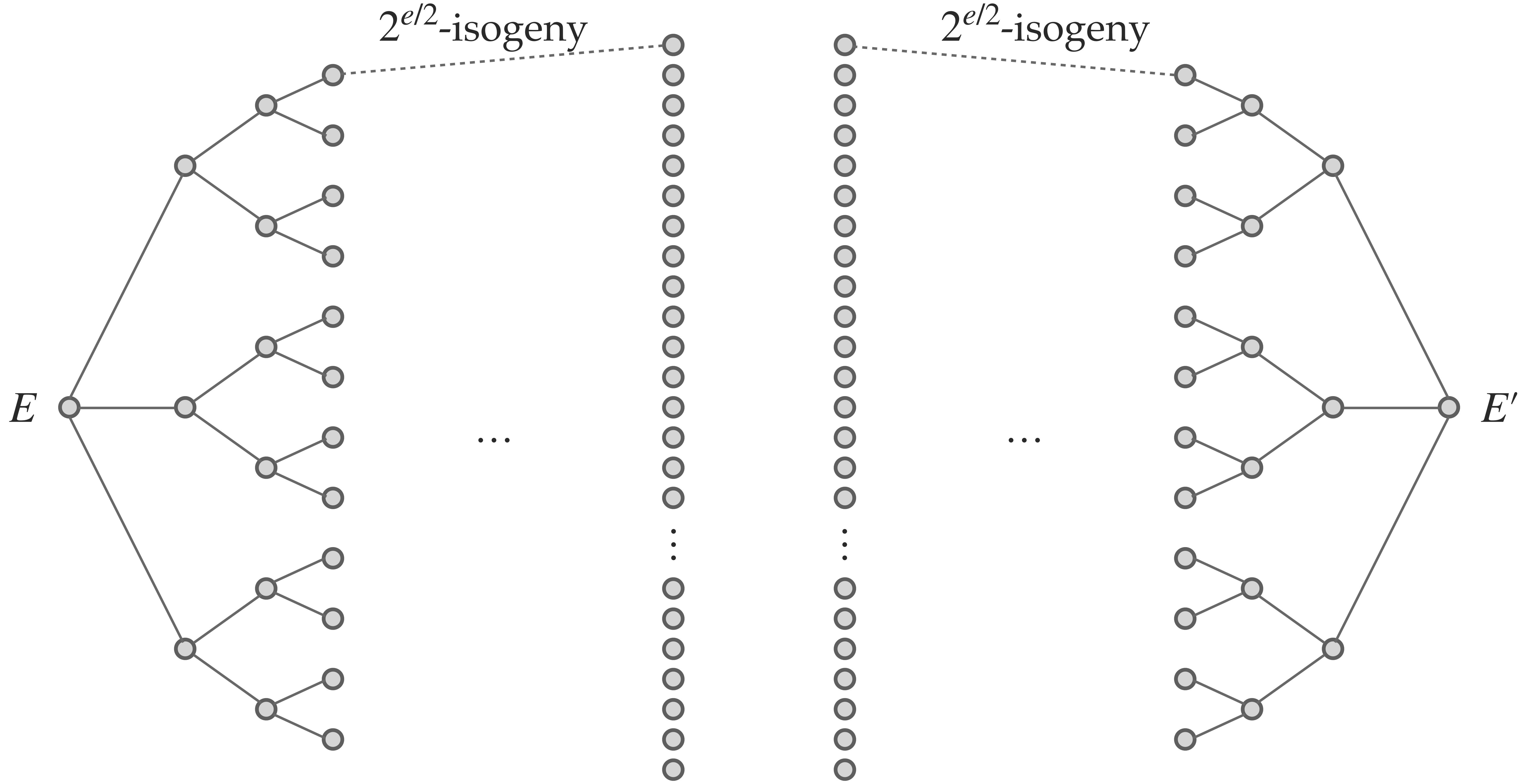
- Meet-in-the-middle
- Parallel Collision Search (vOW)
- Delfs–Galbraith

Meet-in-the-middle

Example. **Goal**: Find a 2^e-isogeny from E to E'.



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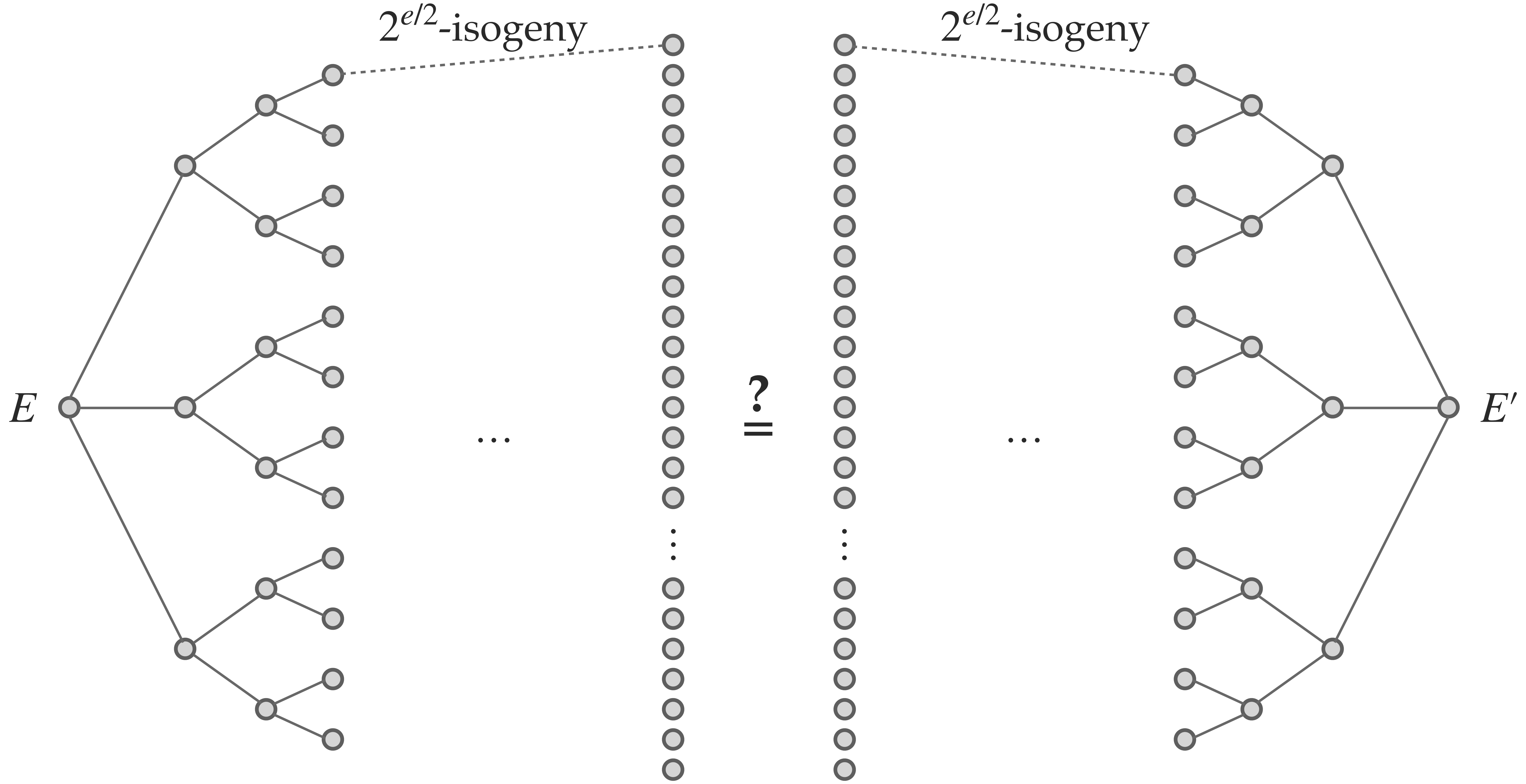

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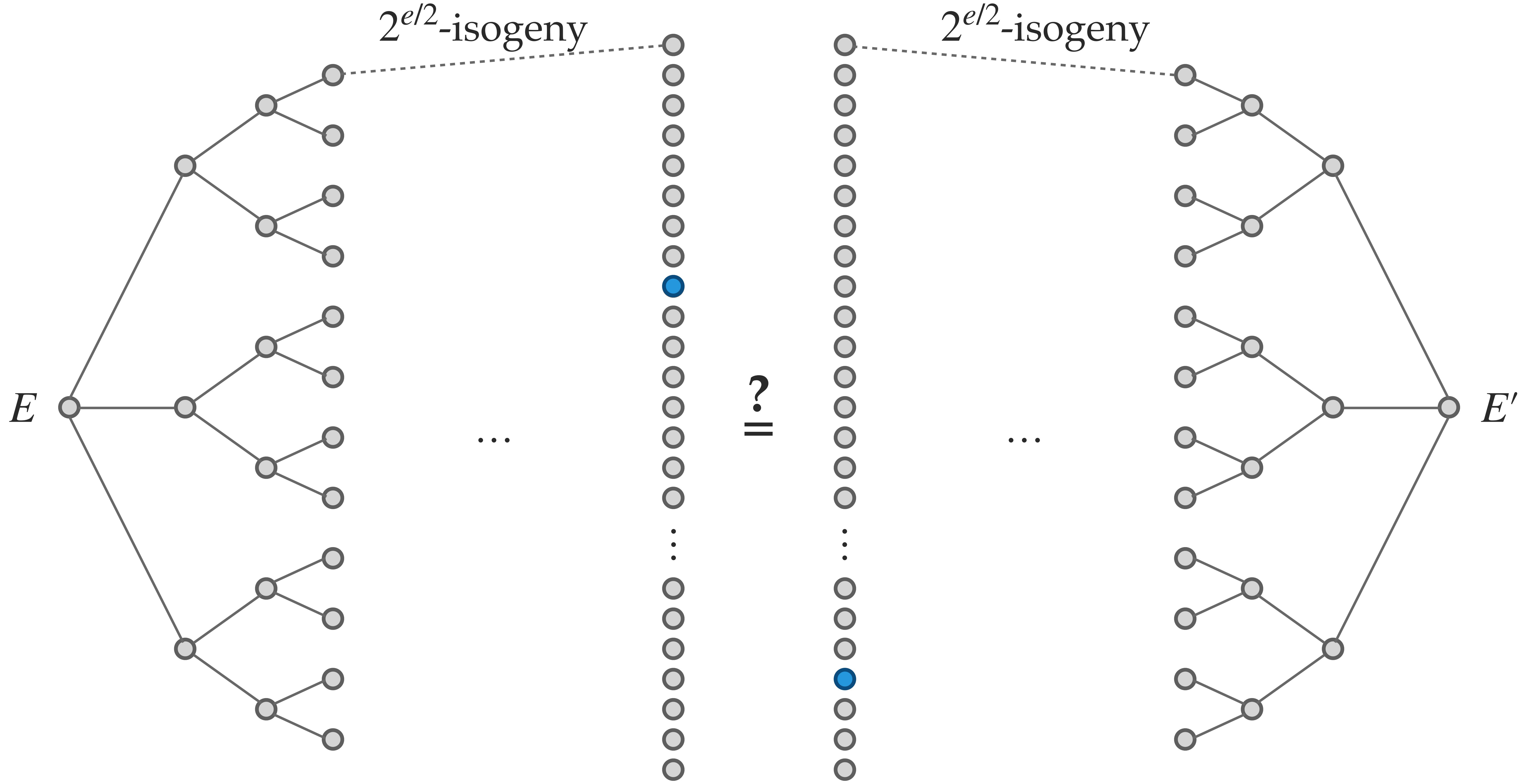

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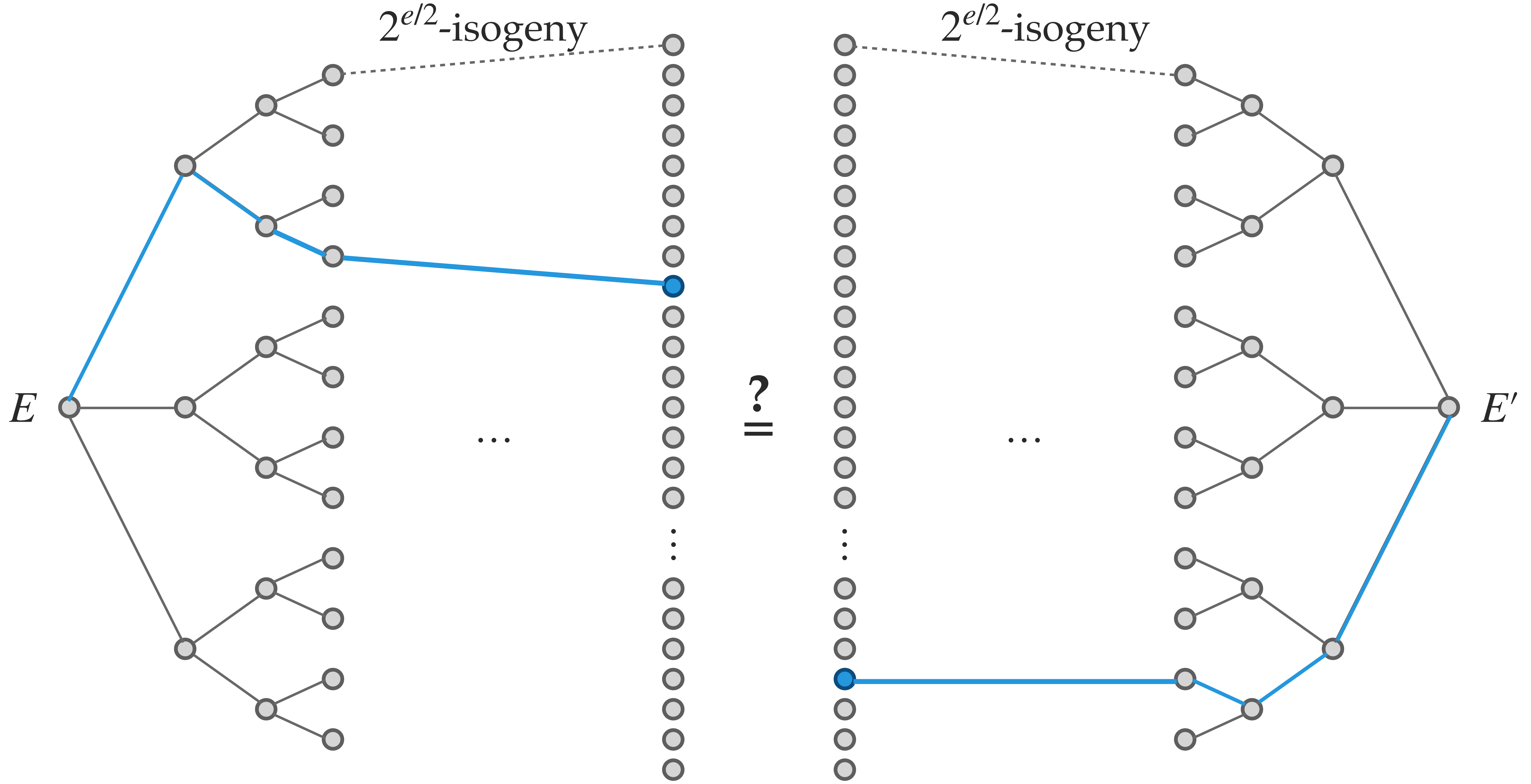

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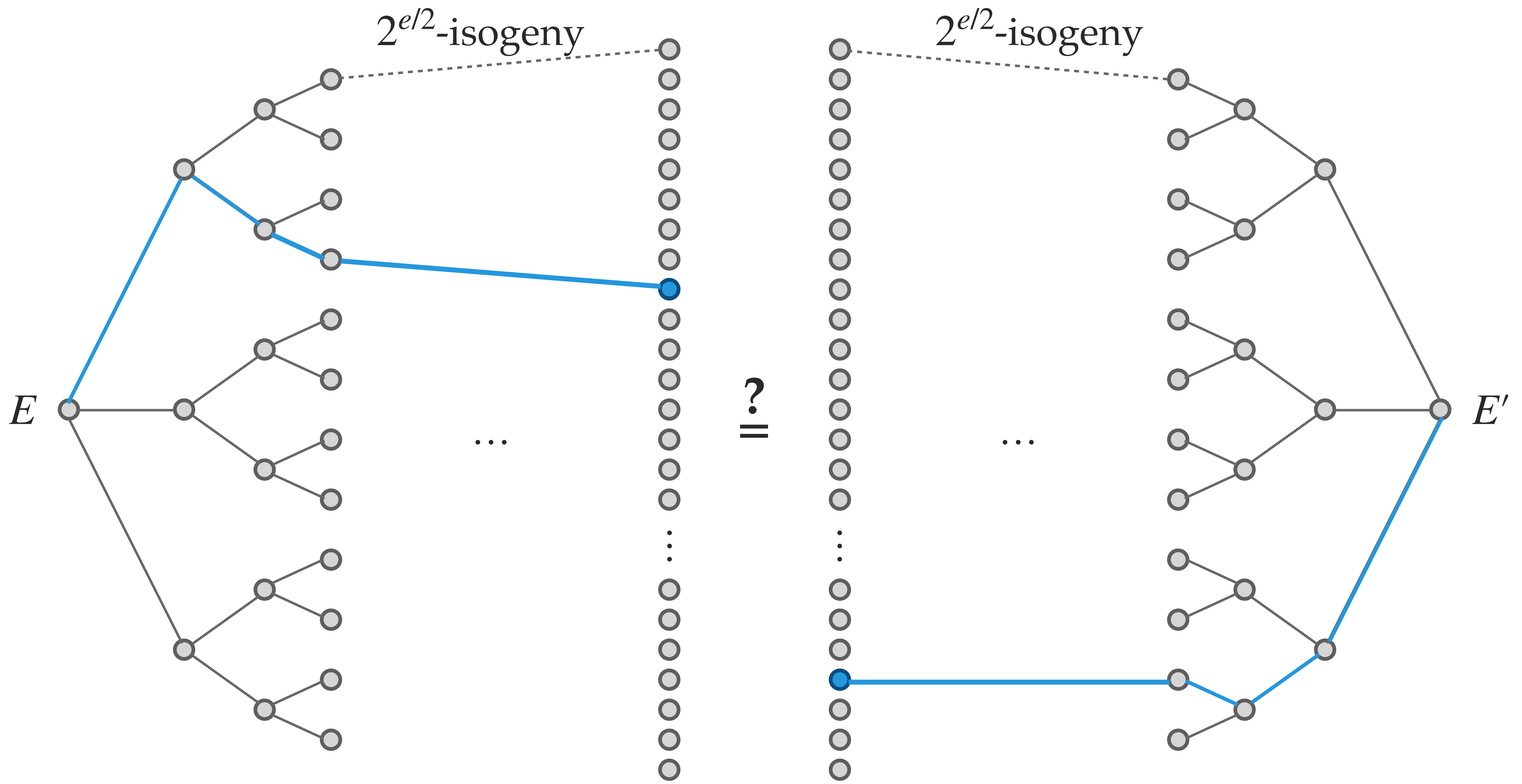

More details in the assignment.

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What is a collision? Why does a collision help us solve the (EC)DLP?

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and we compute

$$x = \frac{a - a'}{b' - b} \pmod{N}.$$

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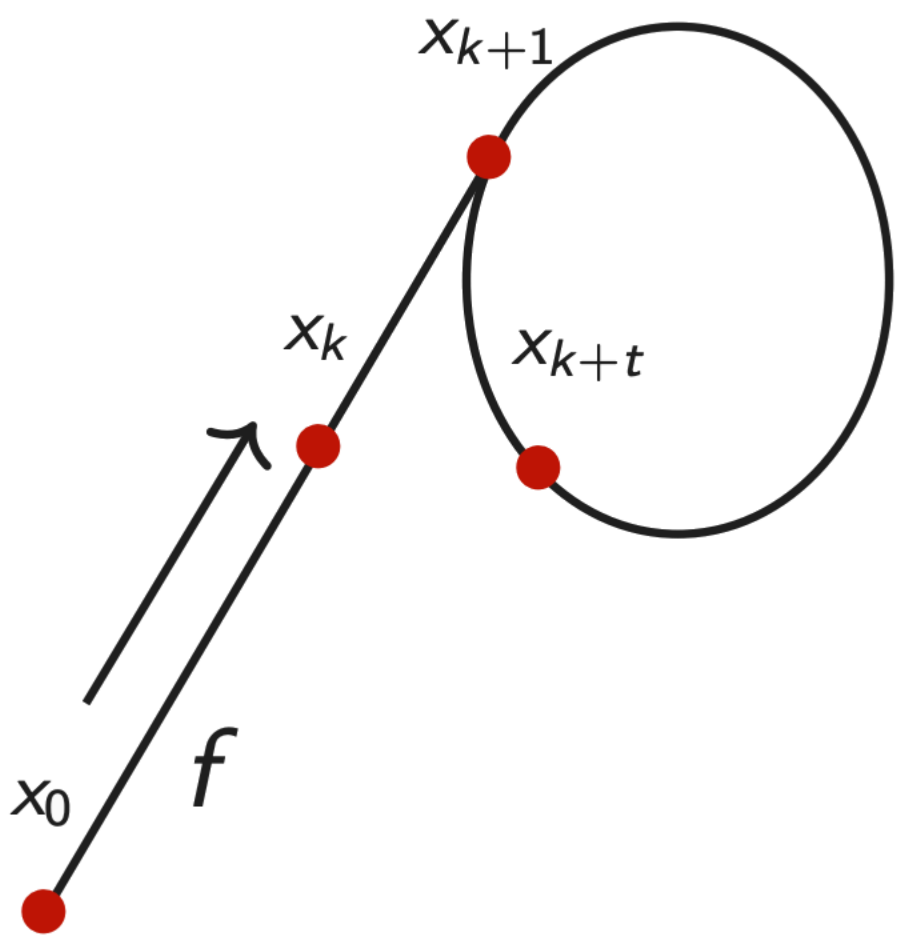

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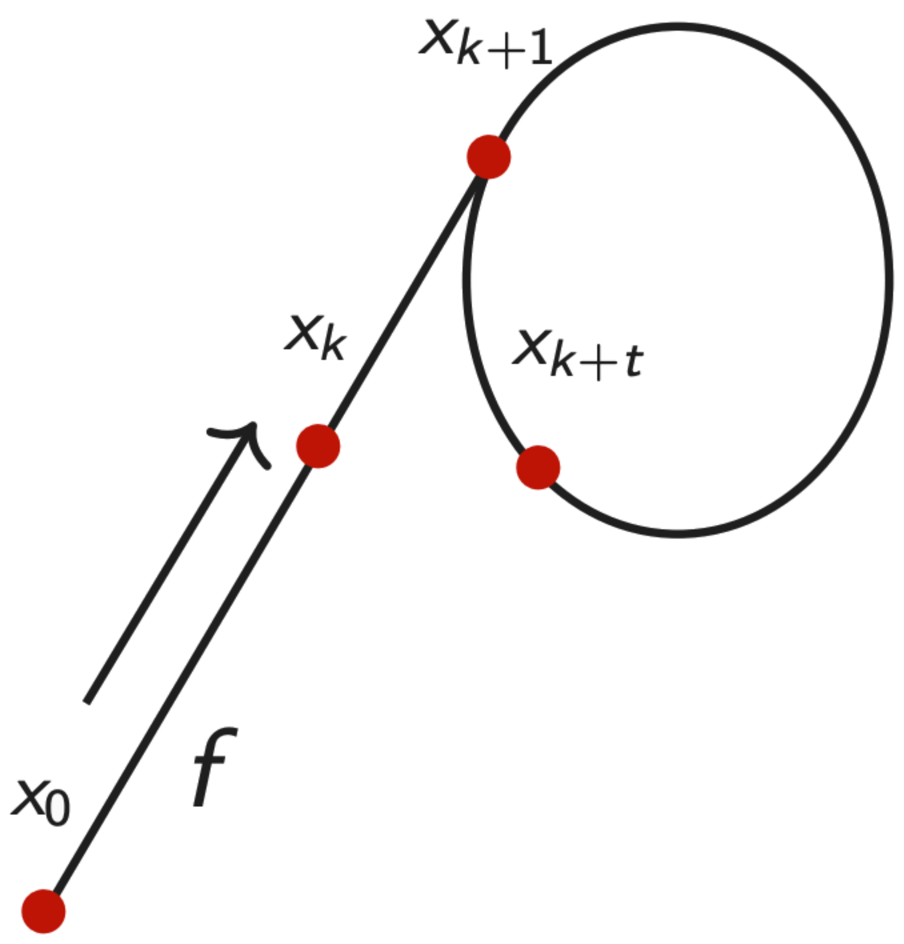

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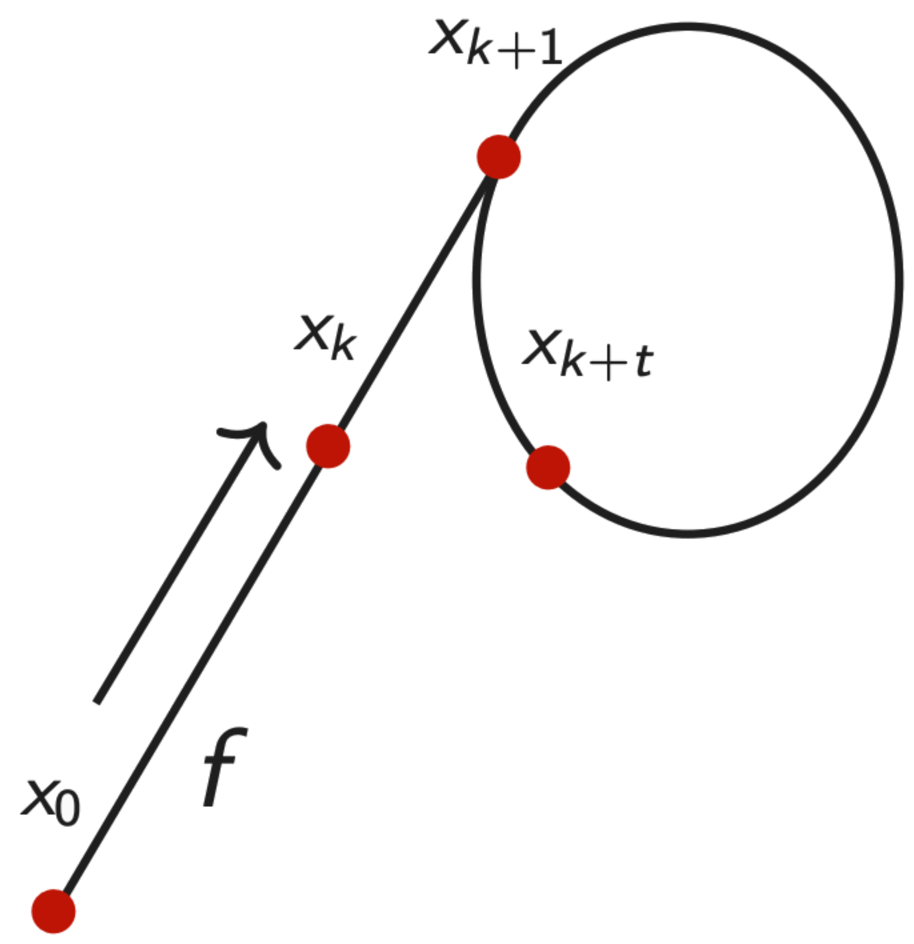

- Ideally, f is a random mapping.
- Expected number of steps until the collision is found

$$\sqrt{\frac{\pi N}{2}}$$

Collision search

$$f(R) = \begin{cases} R + P & \text{if } R \in S_1 \\ 2R & \text{if } R \in S_2 \\ R + Q & \text{if } R \in S_3, \end{cases}$$

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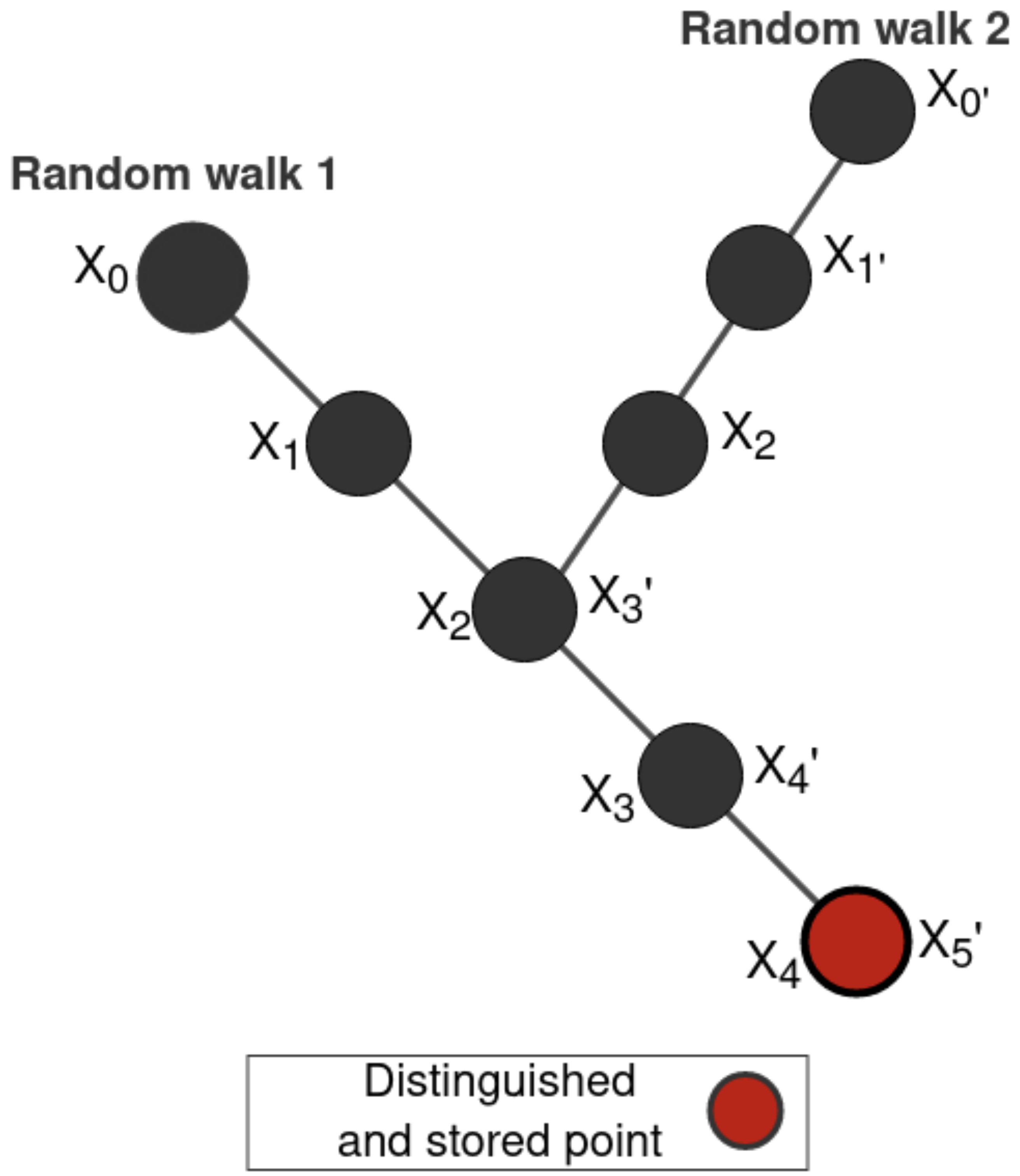
Intuitively:

- Start from $R = aP + bQ$ for some random a and b.
- *Walk* the random walk until we find the same point twice.
  $\hookrightarrow$ To discover the collision, we need to store *all** the points that we compute.

Parallel Collision Search



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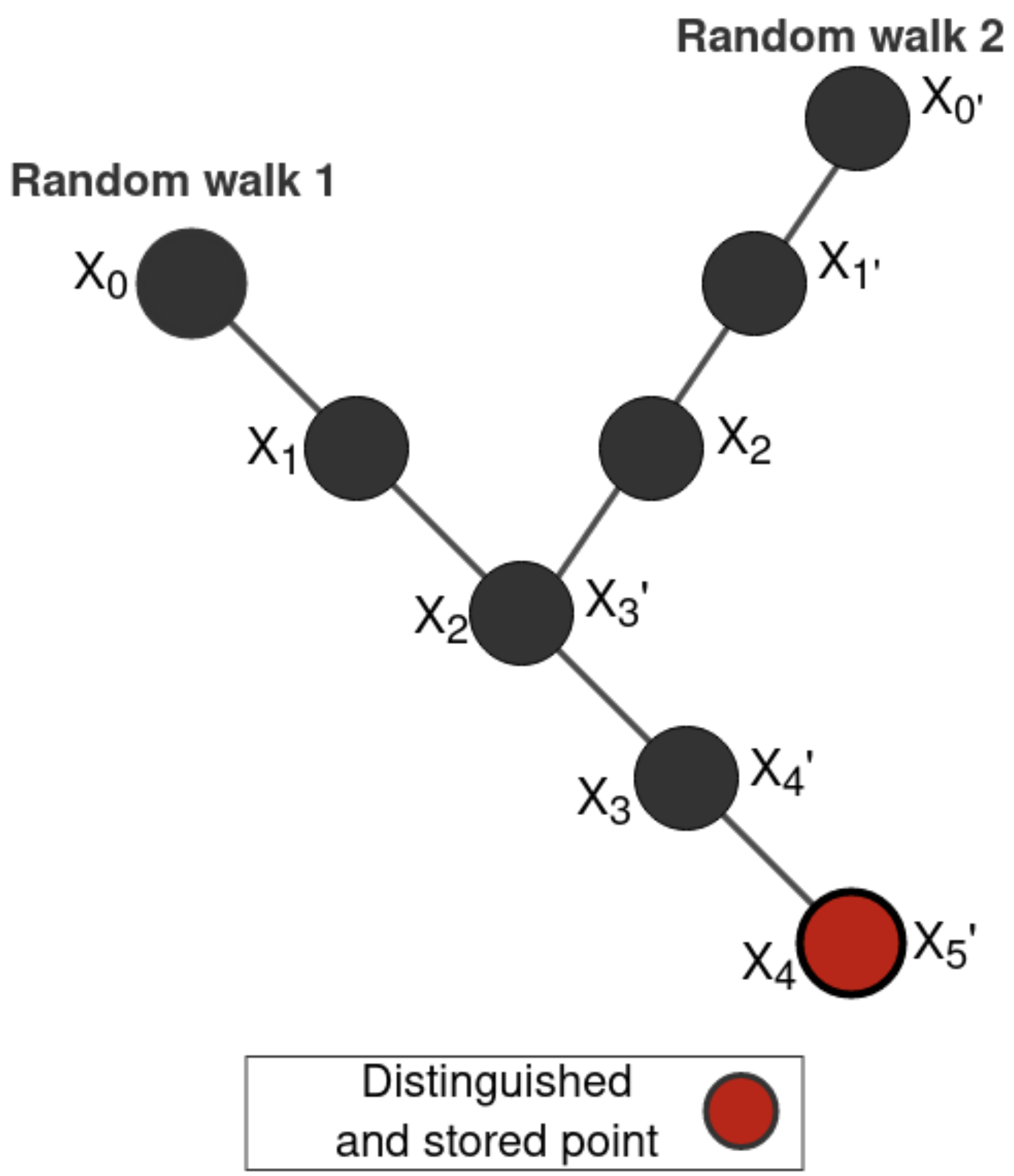

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  ex. The *x*-coordinate has 3 trailling zero bits:
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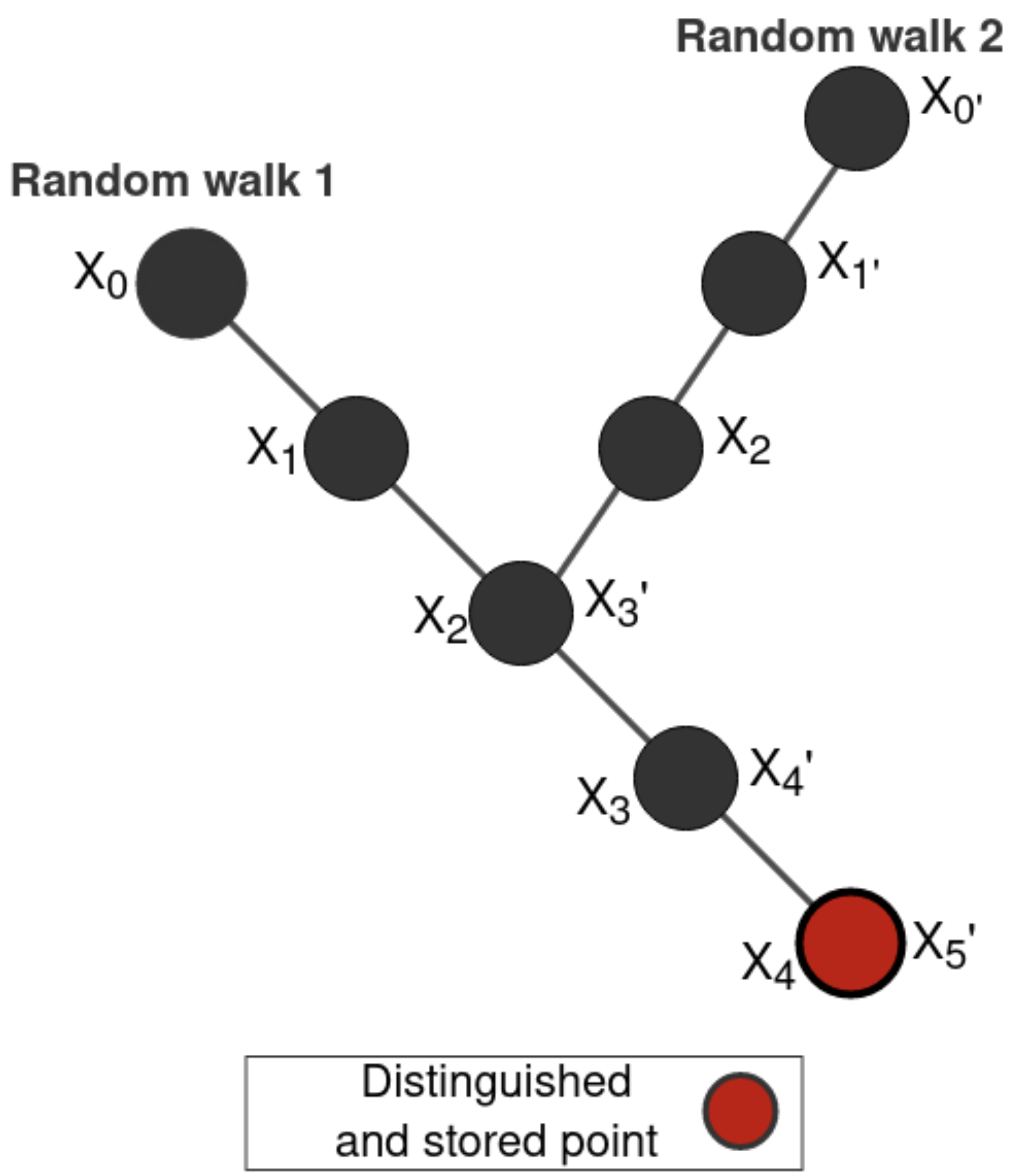

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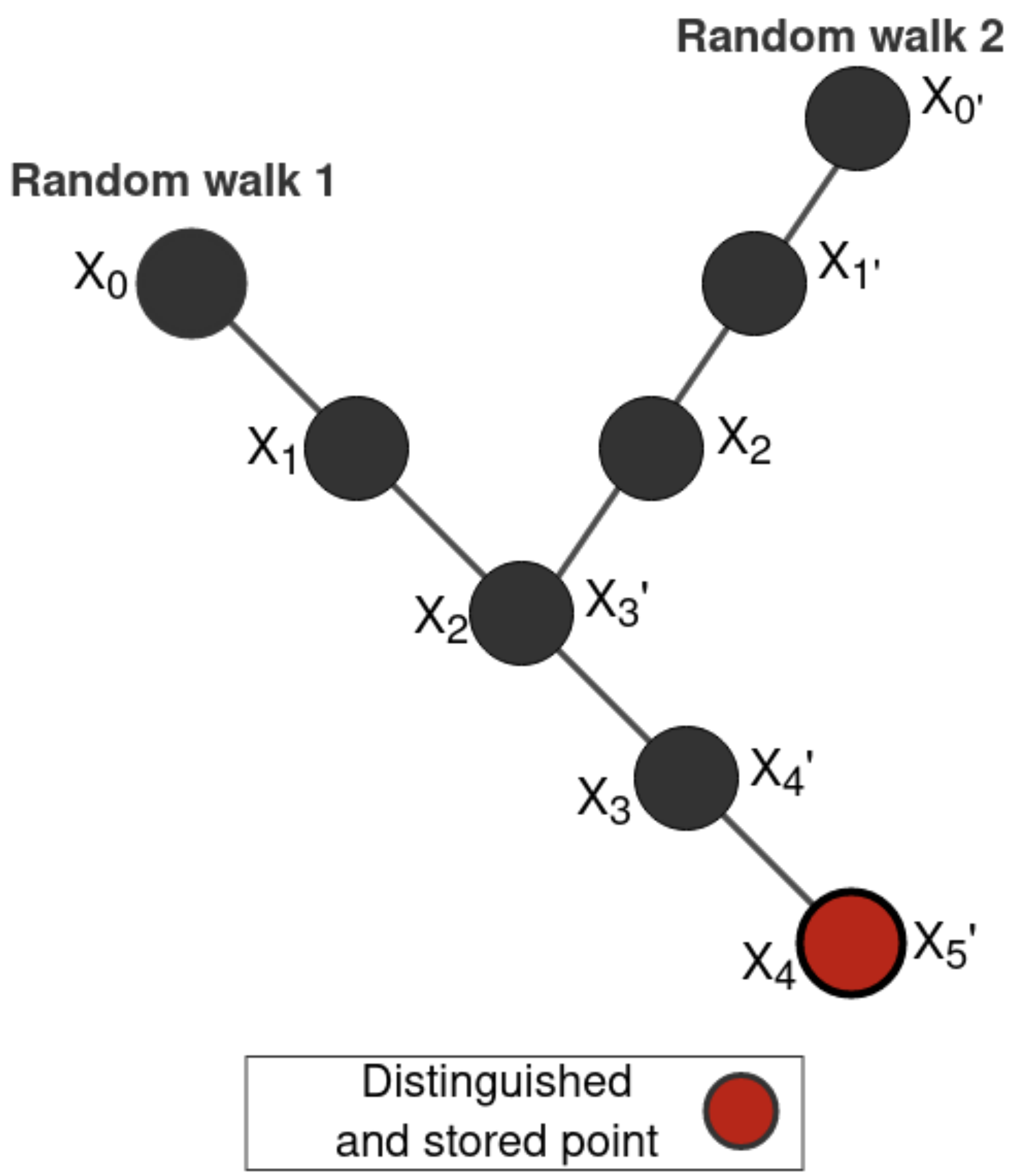

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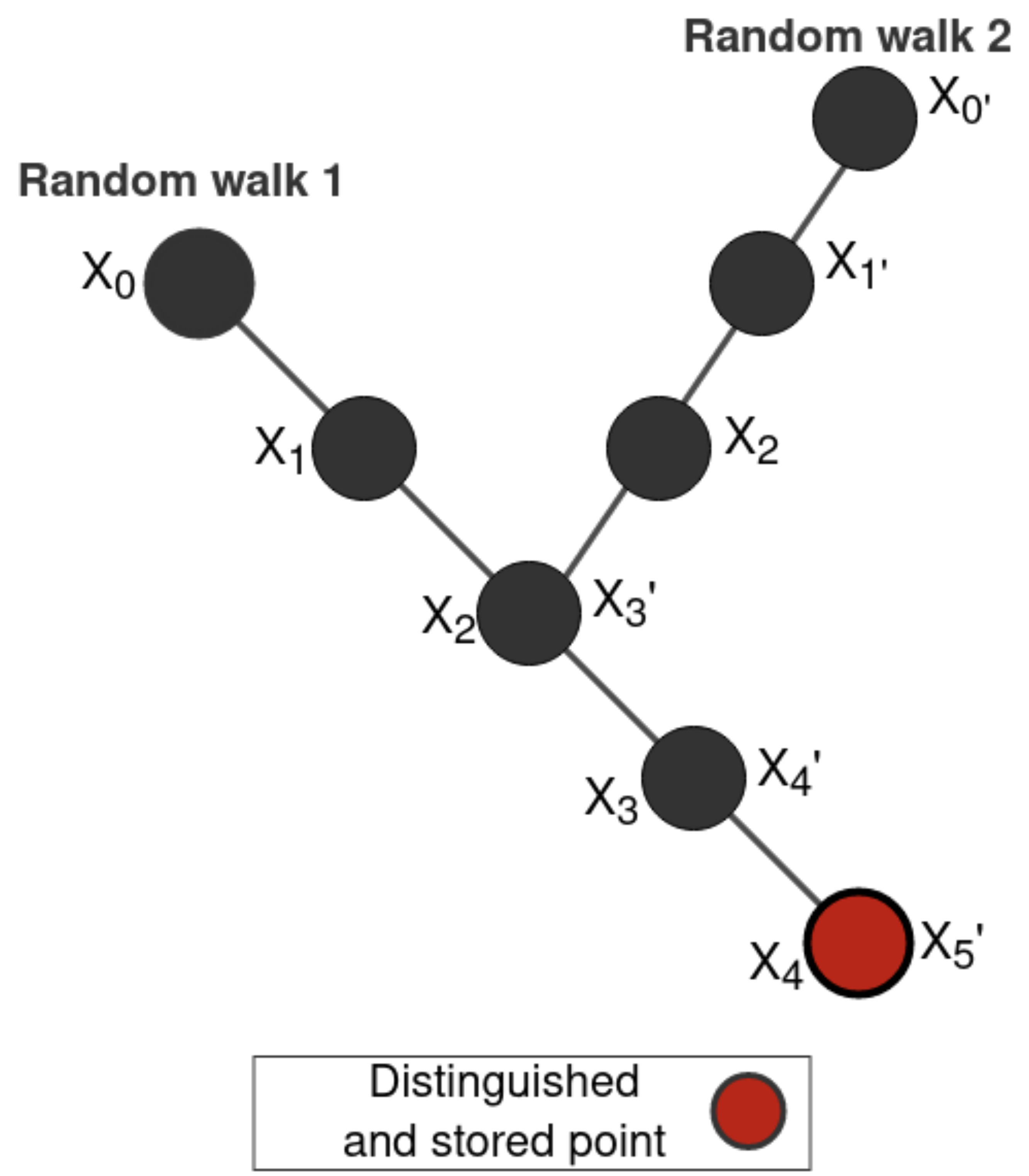

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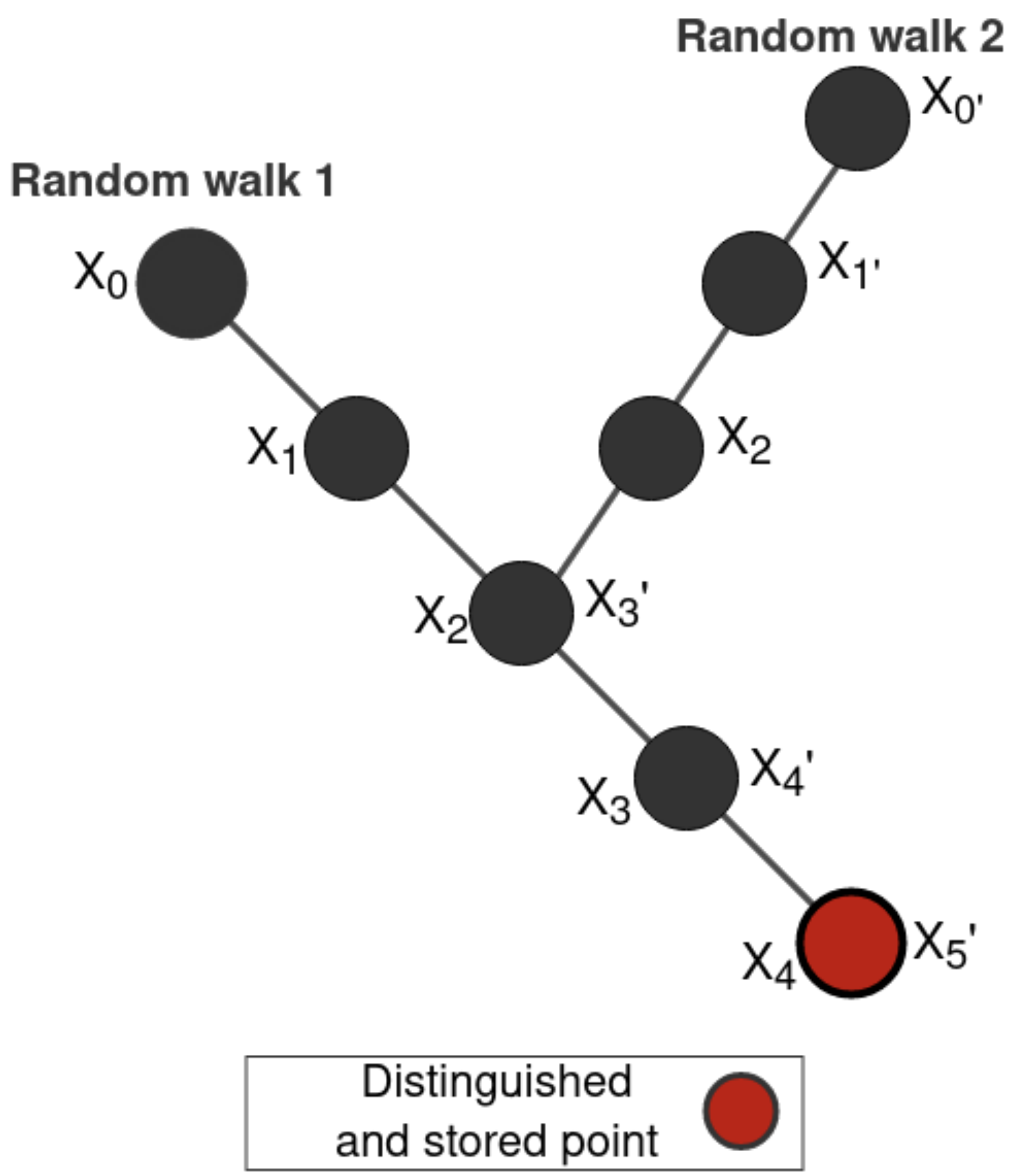

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The Birthday paradox

(recall) $\sim \sqrt{N}$

PCS for isogenies

Yes, but it becomes a multi-collision search (finding the golden collision).

Even less memory

Delfs–Galbraith algorithm.

- Only for the isogeny setting.
- Negligible space requirements.

Building crypto from elliptic curves (not PQ)

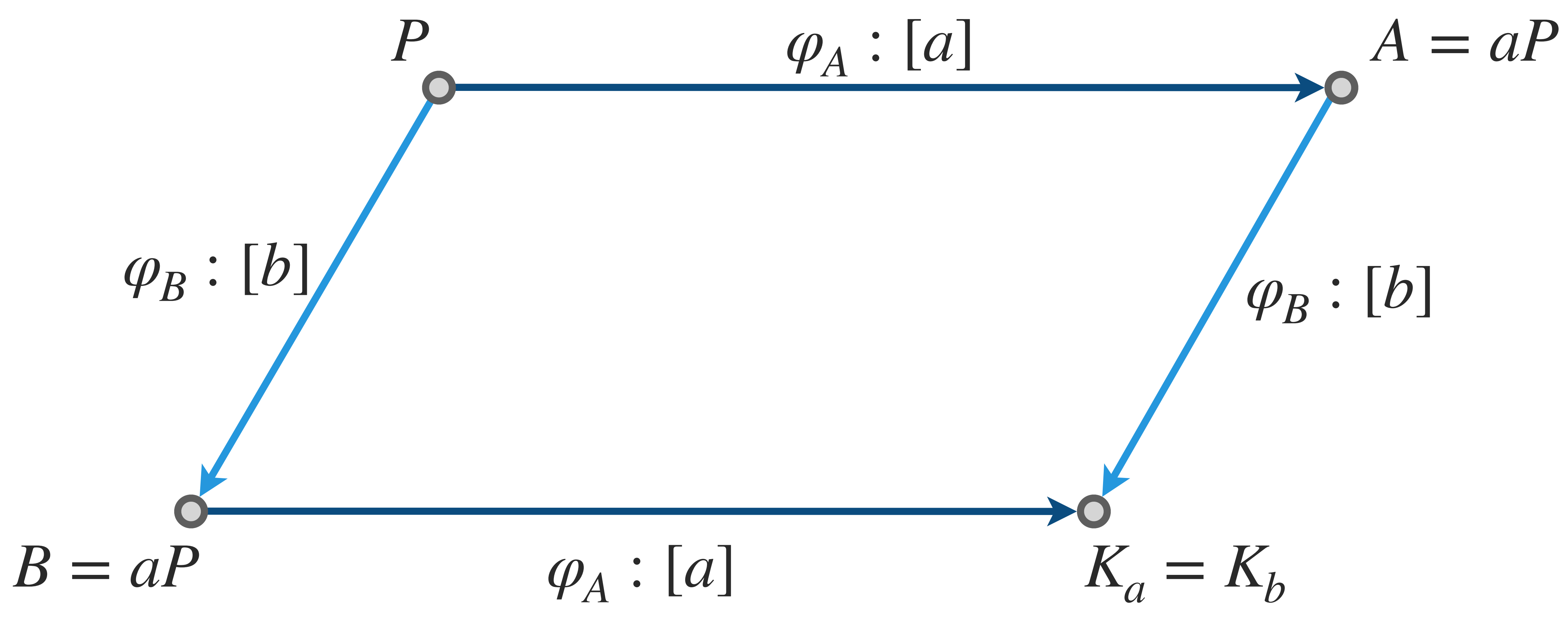
Building crypto from ~~elliptic curves~~ isogenies (PQ)

CSIDH

DH key exchange on graphs

Imagine *the dlog graph*

- Vertices are points on E.
- Edges are multiplication-by-i maps.



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Isogeny graphs

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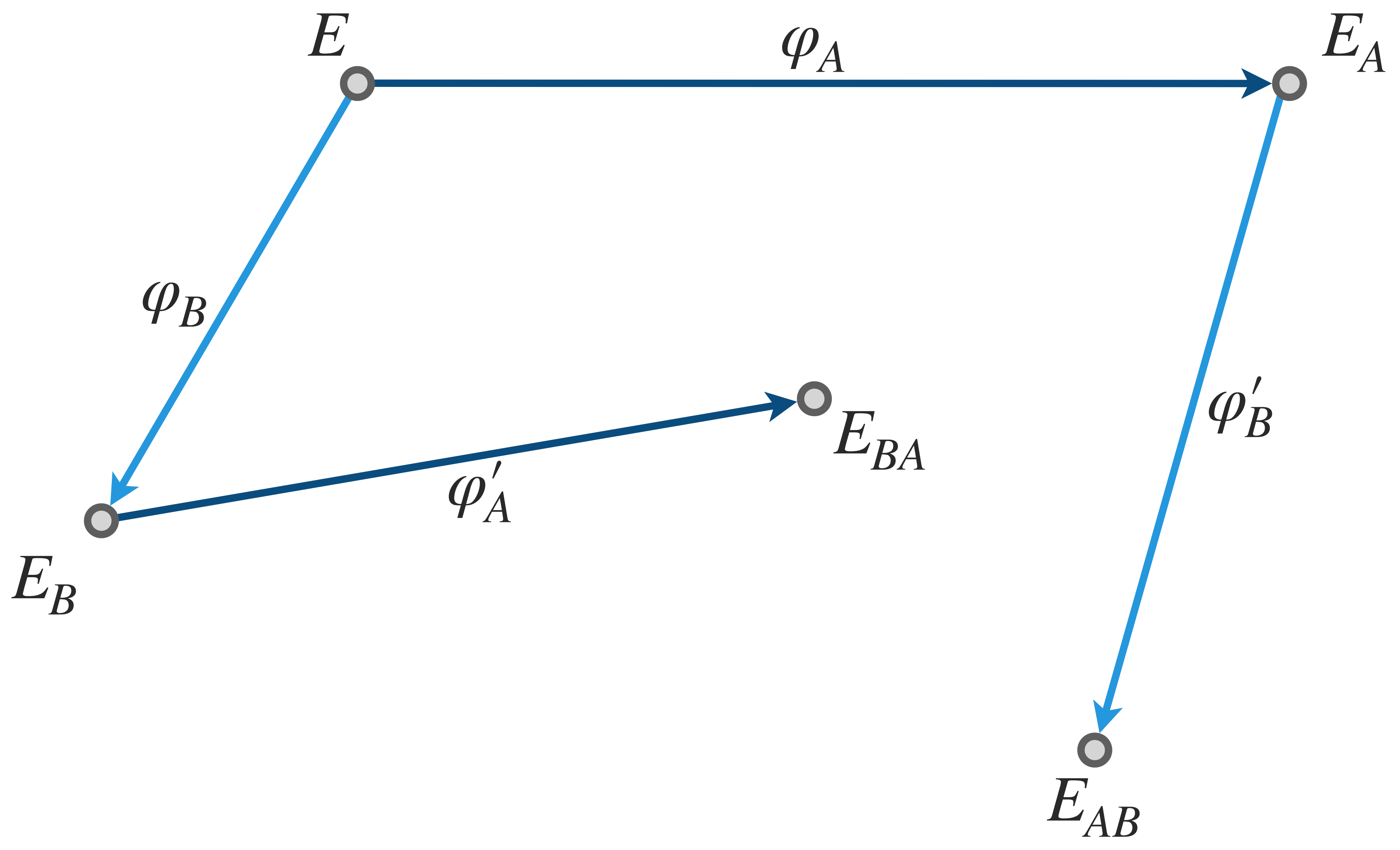

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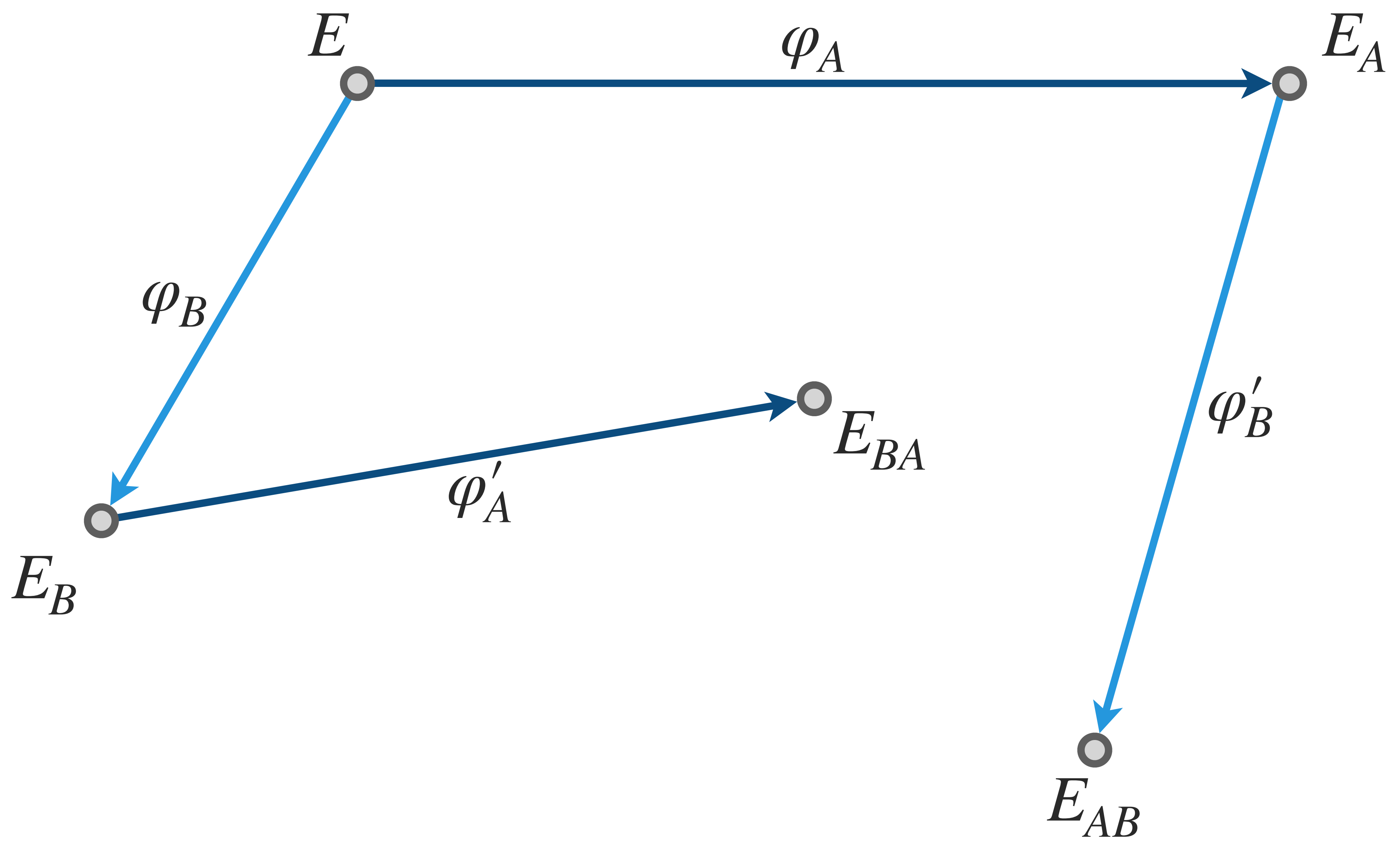

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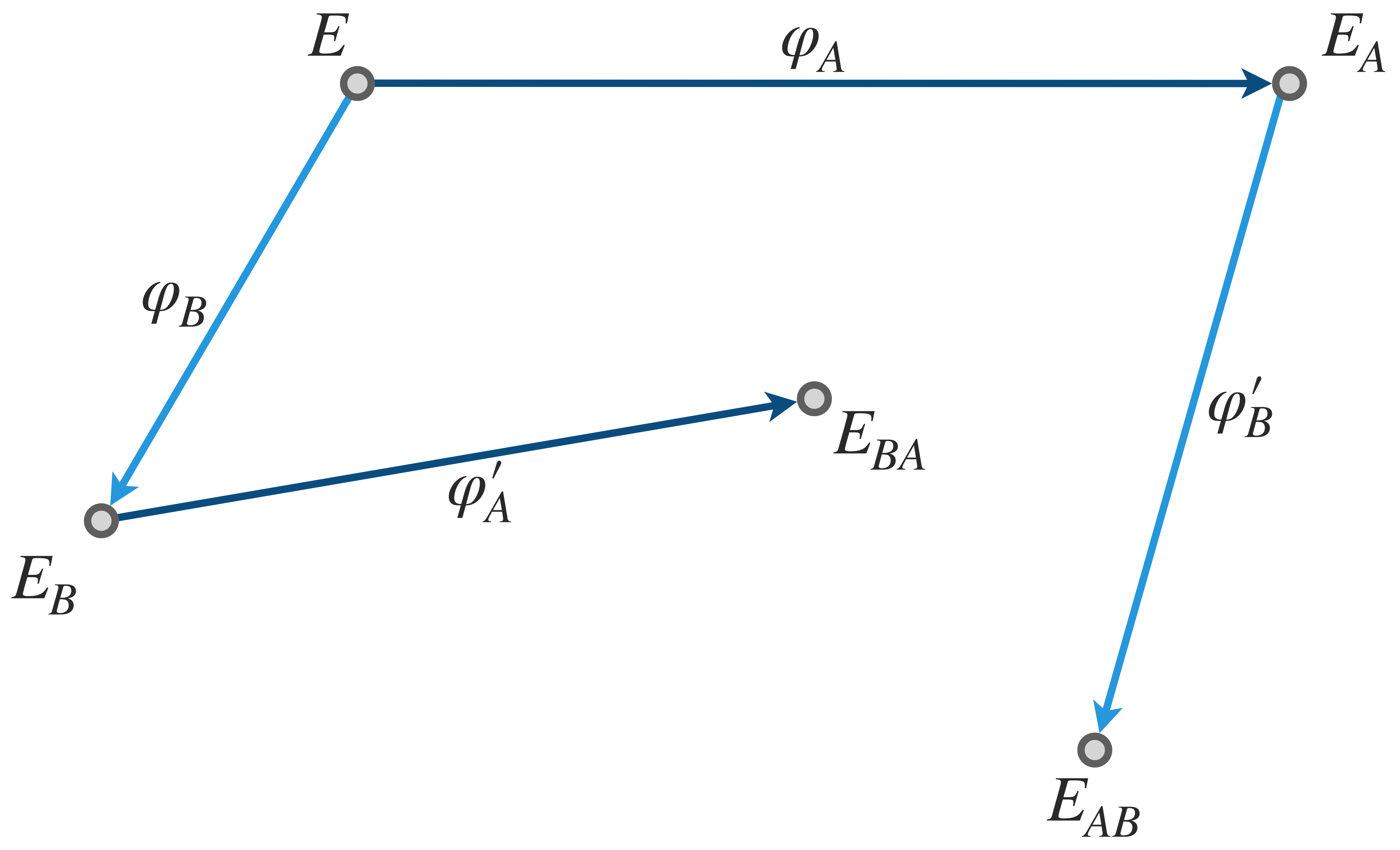

→ Walking on the isogeny graph is not commutative (a priori).

↳ Alice & Bob do not end up on the same vertex (isomorphism class).

Commutative group action

Fundamental theorem of cyclic groups *additive notation.

Every subgroup of a cyclic group $G = \langle P \rangle$ is cyclic.
Moreover, if $\#G = N$, then the order of any subgroup of G is a divisor of N, and,
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Supersingular curves and cyclic groups (recall)

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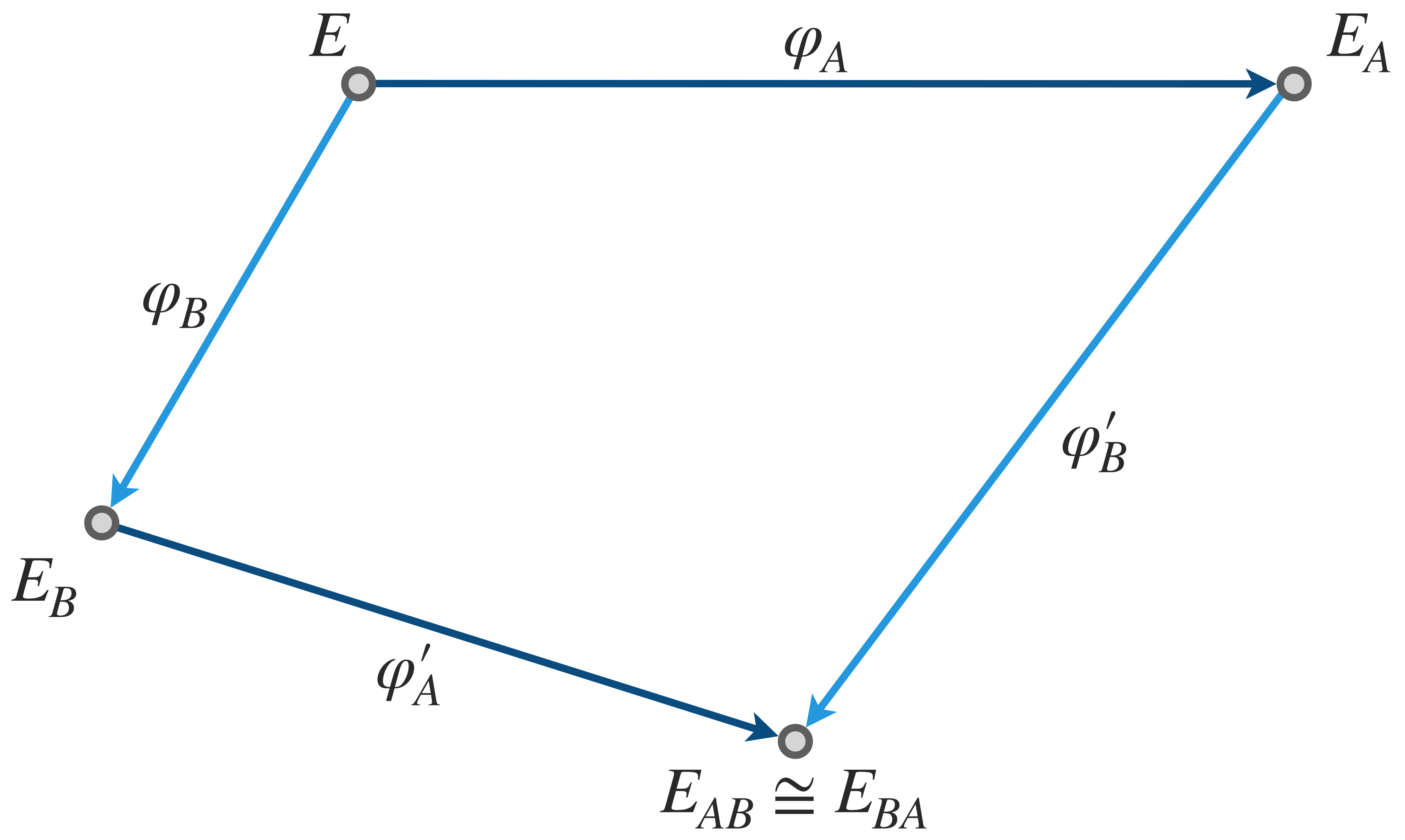
Taking the $E(\mathbb{F}_p)$ isogeny graph will give us a commutative group action.

DH key exchange on isogeny graphs

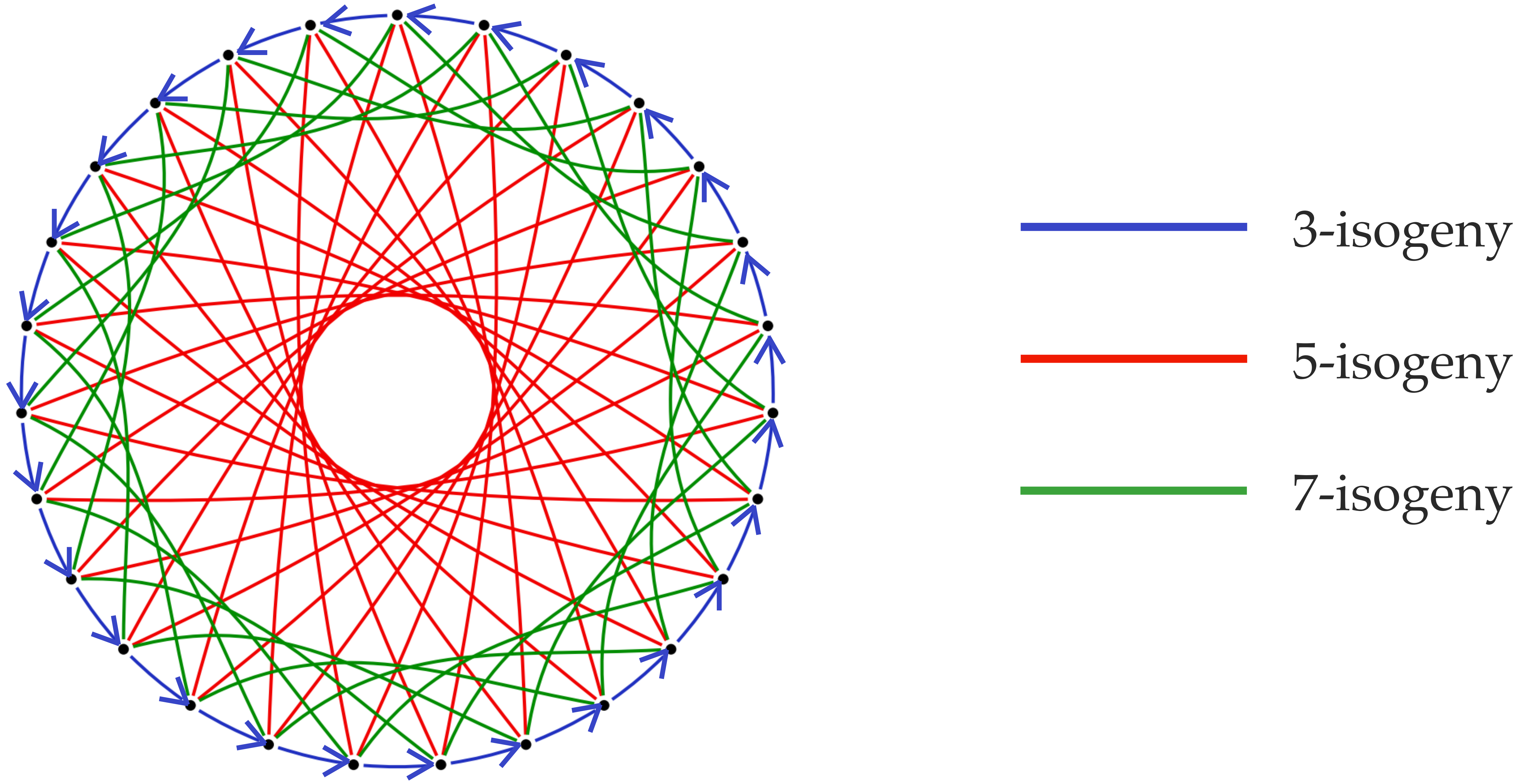
Isogeny graphs $E(\mathbb{F}_p)$ with $p = 4 \cdot \ell_1 \cdots \ell_n - 1$ a prime.

- Vertices are $\mathbb{F}_p$-isomorphism classes of supersingular elliptic curves.
- Edges are prime-degree isogenies between them.



The CSIDH graph

Example. Let $p = 4 \cdot 3 \cdot 5 \cdot 7 - 1$.



Quadratic twists

E'/k is a twist of elliptic curve E/k if E' is isomorphic to E over $\bar{k}$.

For $E : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_p$ with $p \equiv 3 \bmod 4$
$E' : -y^2 = x^3 + Ax^2 + x$ is isomorphic to E via

$$(x, y) \mapsto (x, \sqrt{-1}y).$$

This map is defined over $\mathbb{F}_{p^2}$, so this is a quadratic twist.

E' is not in Weierstrass form (does not have the right shape).
E' is isomorphic to $E'' : y^2 = x^3 - Ax^2 + x$ via $(x, y) \mapsto (-x, y)$ over $\mathbb{F}_p$.

Each $x \in \mathbb{F}_p$ satisfies one of

- $x^3 + Ax^2 + x$ is a square in $\mathbb{F}_p$, thus there are two points $(x, \pm\sqrt{x^3 + Ax^2 + x})$ in $E(\mathbb{F}_p)$.
- $x^3 + Ax^2 + x$ is not a square in $\mathbb{F}_p$, thus there are two points $(x, \pm\sqrt{-(x^3 + Ax^2 + x)})$ in $E'(\mathbb{F}_p)$.
- $x^3 + Ax^2 + x = 0$, thus $(x, 0)$ is a point in $E(\mathbb{F}_p)$ and in $E'(\mathbb{F}_p)$.

Quadratic twists in SageMath

Quadratic twists

```
p=419
Fp=GF(p)
Fp2=GF(p^2)
E=EllipticCurve(Fp, [0, 410, 0, 1, 0])
assert E.order()==p+1 #check that it is a supersingular curve
E_t=E.quadratic_twist()
print("The quadratic twist of E_", E.montgomery_model().a2(), "is E_", E_t.
montgomery_model().a2())
print("Indeed, -", E.montgomery_model().a2(), "is ", -Fp(E.montgomery_model().a2
()), "over ", Fp)
```

✓ 0.0s

```
The quadratic twist of E_ 410 is E_ 9
Indeed, - 410 is  9 over  Finite Field of size 419
```

The CSIDH graph

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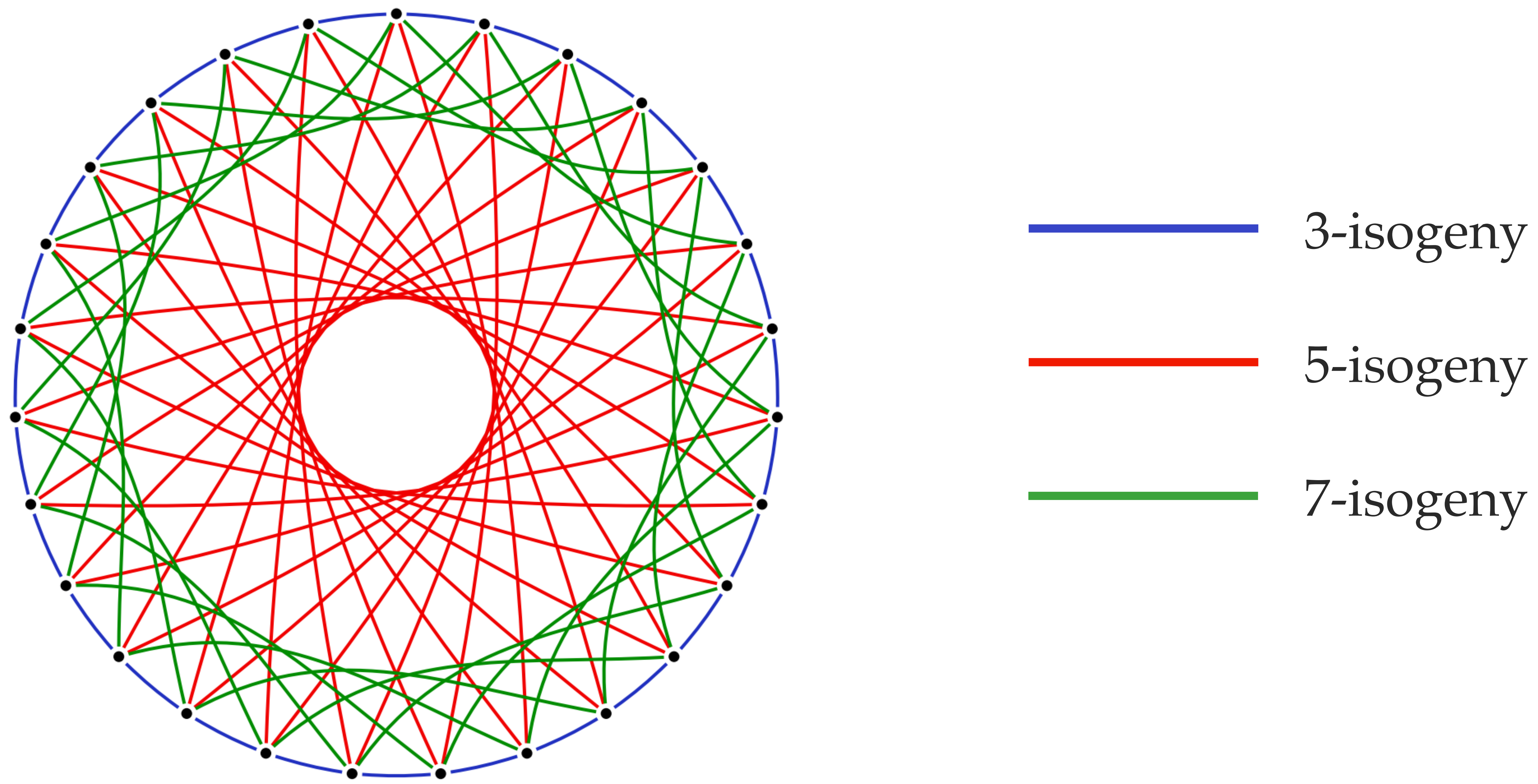

CSIDH

CSIDH

- Choose small primes $\ell_1, \ldots, \ell_n$, making sure $p = 4 \cdot \ell_1 \cdots \ell_n - 1$ is prime → we can compute ℓ_i-steps in the positive or in the negative direction, for all ℓ_i.

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Example. CSIDH-512: $p = 4 \cdot \prod \ell_i - 1$, for $\ell_i \in \{3,5,\ldots,377,587\}$ (the first 73 primes and 587).

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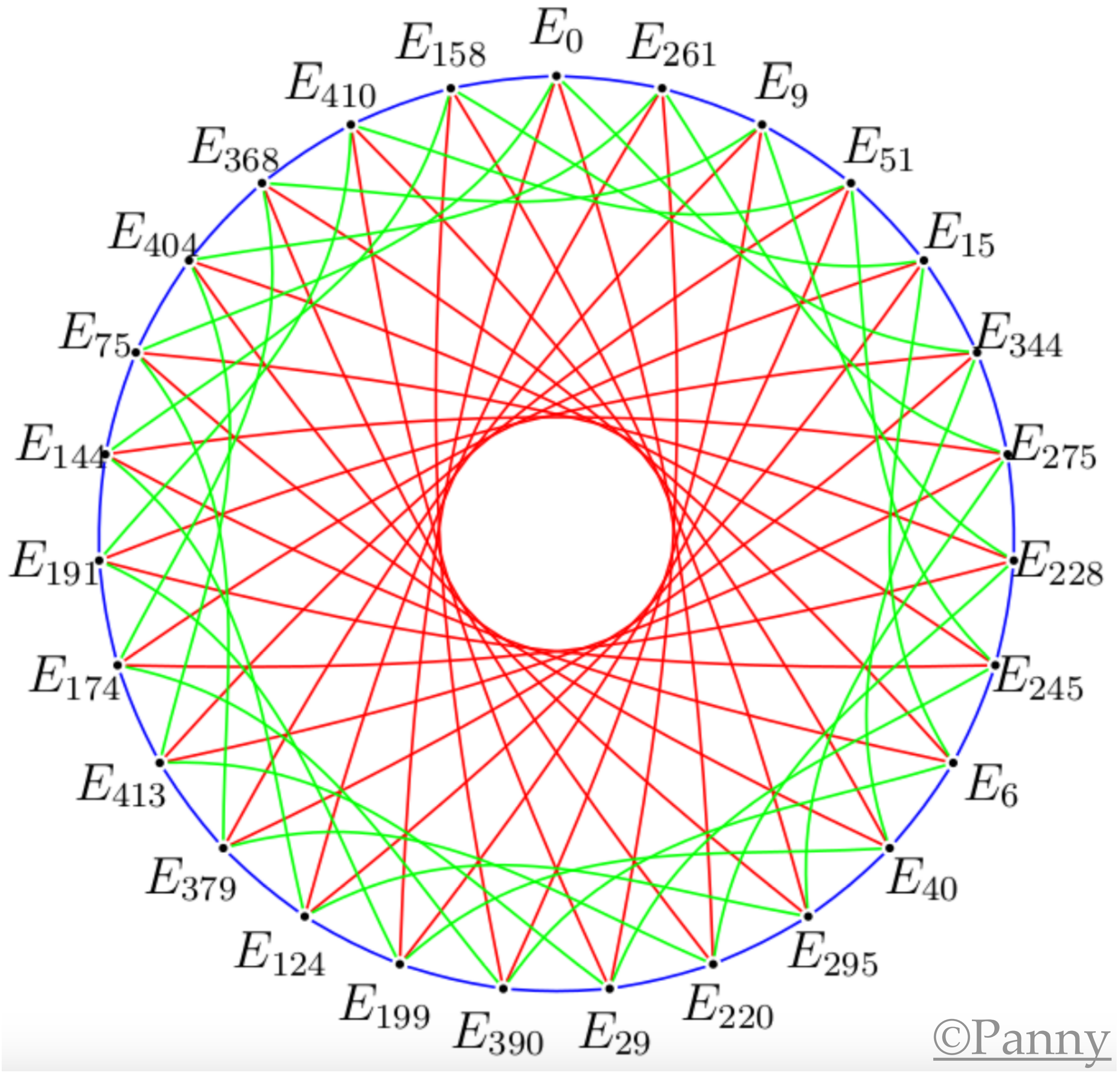

Example. $p = 419 = 4 \cdot 3 \cdot 5 \cdot 7 - 1$.

Walking the CSIDH graph

Taking a positive ℓ_i-step.

Taking a negative ℓ_i-step.

Walking the CSIDH graph

Taking a positive ℓ_i-step.

- Find a point $(x, y) \in E$ of order ℓ_i with $x, y \in \mathbb{F}_p$.
- Compute the isogeny with kernel $\langle (x, y) \rangle$ using Vélu's formulas.

Taking a negative ℓ_i-step.

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Taking a negative ℓ_i-step.

- Find a point $(x, y) \in E$ of order ℓ_i with $x \in \mathbb{F}_p$, but $y \notin \mathbb{F}_p$.
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Walking the CSIDH graph

Taking a positive ℓ_i-step.

- Find a point $(x, y) \in E$ of order ℓ_i with $x, y \in \mathbb{F}_p$.
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or

- Go to the quadratic twist. Compute a positive ℓ_i-step. Go to the quadratic twist.

What we did not cover

- The history of SIDH.
- Why CSIDH represents an action of an ideal-class group.
- The Deuring correspondence.
- Isogenies in higher dimensions.
- SQISign (intuition in assignment - then ask me in the next tutorial)
- Many emerging schemes.
- …