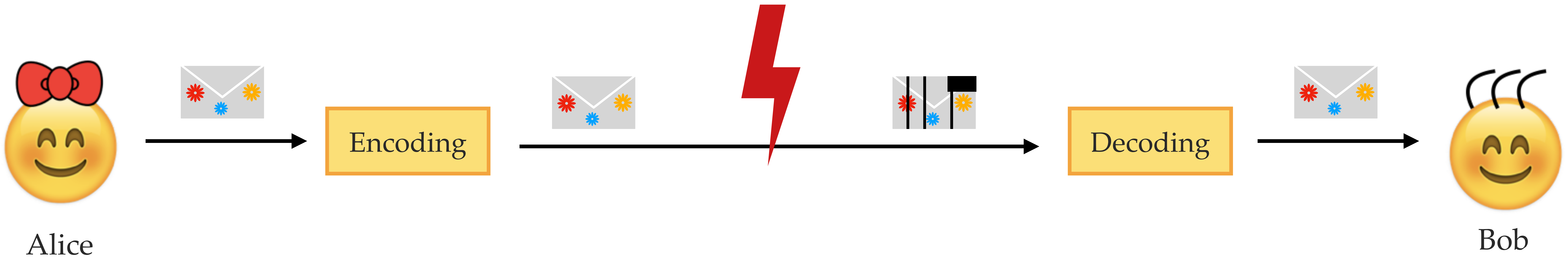
Code-based cryptography II

Monika Trimoska

Selected Areas in Cryptology - Part 1
Spring, 2024

TU/e

Error-correcting codes (recall)



- Primary use case: communication over a noisy channel.
- Main idea: introduce some redundancy in order to be able to correct the errors.
- Some structured error-correcting codes have efficient decoding algorithms.
- Decoding is, in general, a hard problem - so it is hard for *random* codes.

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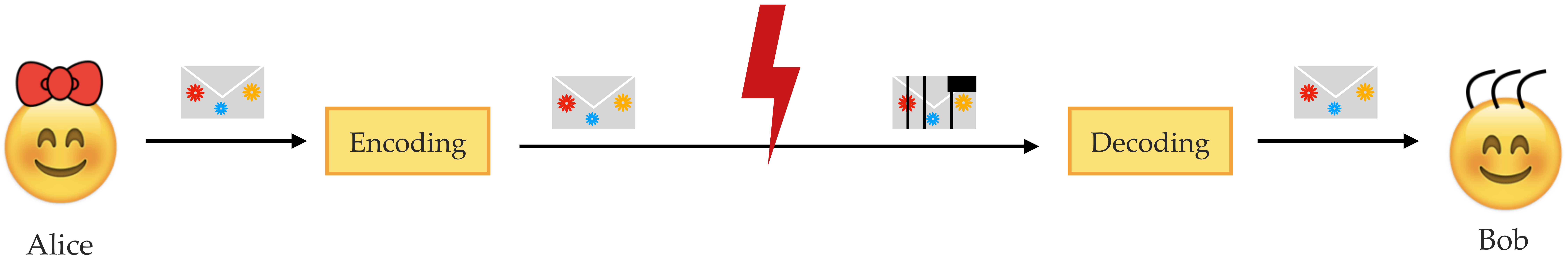

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Hard problems (often) find their use in cryptography.

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Hamming metric

For $\mathbf{x} \in \mathbb{F}_q^n$, the Hamming weight of $\mathbf{x}$ is the number of nonzero elements, aka.

$$\mathrm{wt}(\mathbf{x}) = |\{i \in \{1, \ldots, n\} \,|\, x_i \neq 0\}|.$$

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Generator matrix

The matrix $\mathbf{G} \in \mathbb{F}_q^{k \times n}$ is called a generator matrix of $\mathscr{C}$, if

$$\mathscr{C} = \{\mathbf{x}\mathbf{G} \mid \mathbf{x} \in \mathbb{F}_q^k\}.$$

Binary linear codes

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Example. $q = 2, n = 5, k = 3$

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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Codewords: $\lambda_1(10101) + \lambda_2(11000) + \lambda_3(11110)$

Example. $\mathbf{c}_1 = (111)\mathbf{G} = (10011),$
$\mathbf{c}_2 = (100)\mathbf{G} = (10101)$

Decoding

- Encoding: $\mathbf{c} = \mathbf{mG}$

- Introducing error $\mathbf{e}$ of low weight: $\mathbf{y} = \mathbf{c} + \mathbf{e} = \mathbf{mG} + \mathbf{e}$, s.t. $\mathrm{wt}(\mathbf{e}) = t$.

- Decoding: Given $\mathbf{y}$, find $\mathbf{c}$ s.t. $\mathbf{y} = \mathbf{c} + \mathbf{e}$ and $\mathrm{wt}(\mathbf{e}) \leq t$.

Representations of linear codes

→ The row space of a generator matrix $\mathbf{G} \in \mathbb{F}_2^{k \times n}$:

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Example. $n = 7, k = 4$

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From **G** to **H**

From $\mathbf{G}$ to $\mathbf{H}$

Systematic form

A systematic generator matrix is a generator matrix of the form $(\mathbf{I}_k \,|\, \mathbf{Q})$, where $\mathbf{I}_k$ is the $k \times k$ identity matrix and $\mathbf{Q}$ is a $k \times (n-k)$ matrix.

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Example. $\mathbf{G} \to \tilde{\mathbf{G}}$

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$\longrightarrow$ When $\mathbf{c} = \mathbf{m}\tilde{\mathbf{G}}$, the first k positions of $\mathbf{c}$ are $\mathbf{m}$.

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Example. $(1010)\tilde{\mathbf{G}} = (1010101)$

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Example: Hamming code

Columns correspond to a bit pattern of length $(n - k)$.

Example. $n = 7, k = 4$

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An error occurs.

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The failure pattern uniquely identifies the error location.

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We will call it a syndrome.

Syndrome decoding

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The syndrome depends only on the error vector.

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Find $\mathbf{e}$ of minimum weight.

Information set decoding

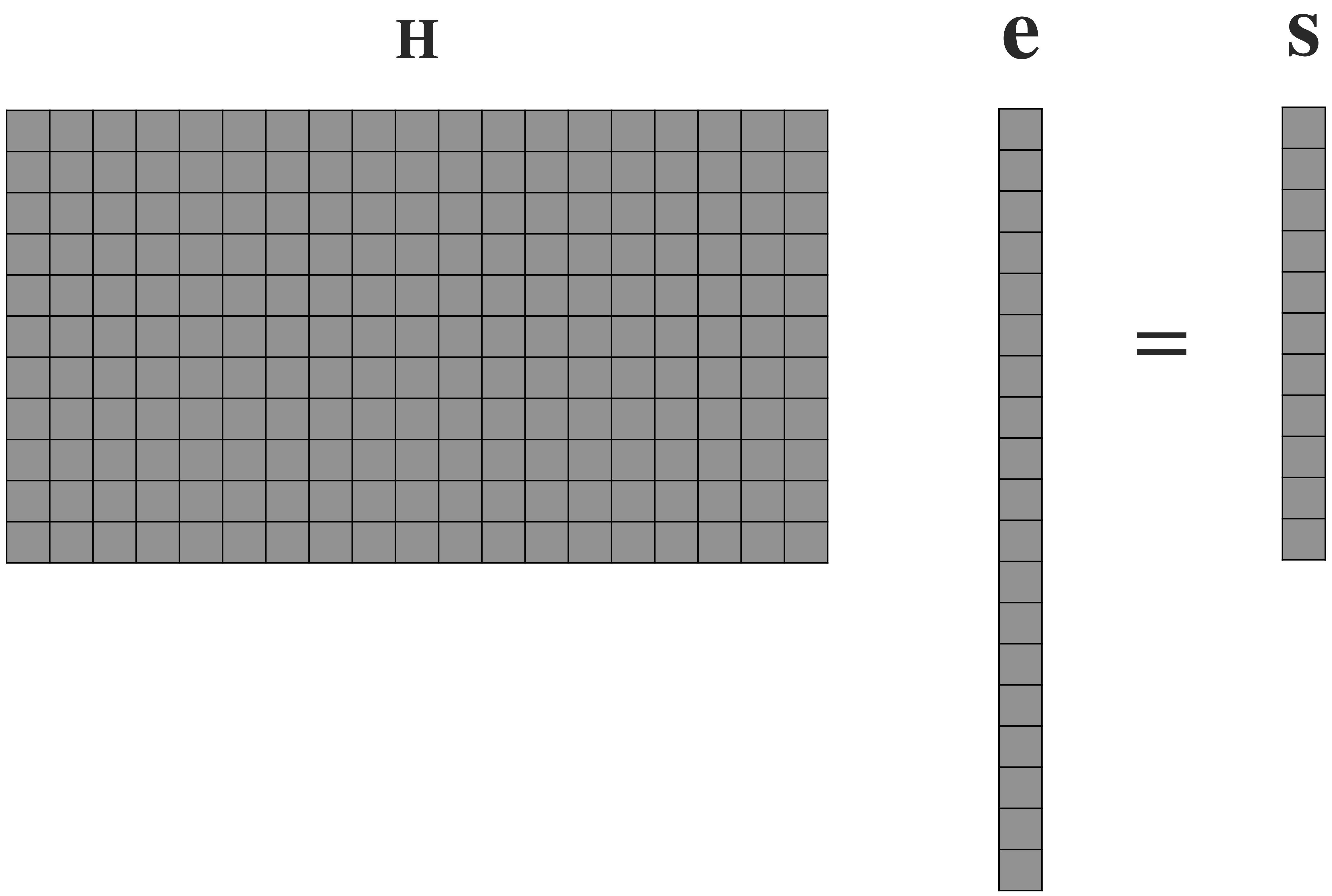
Information set decoding algorithms

Focus on the case $\mathrm{wt}(\mathbf{e}) = t$.

The syndrome decoding problem

Given a syndrome $\mathbf{s} = \mathbf{He}$, find $\mathbf{e}$ such that $\mathrm{wt}(\mathbf{e}) = t$.

Brute force



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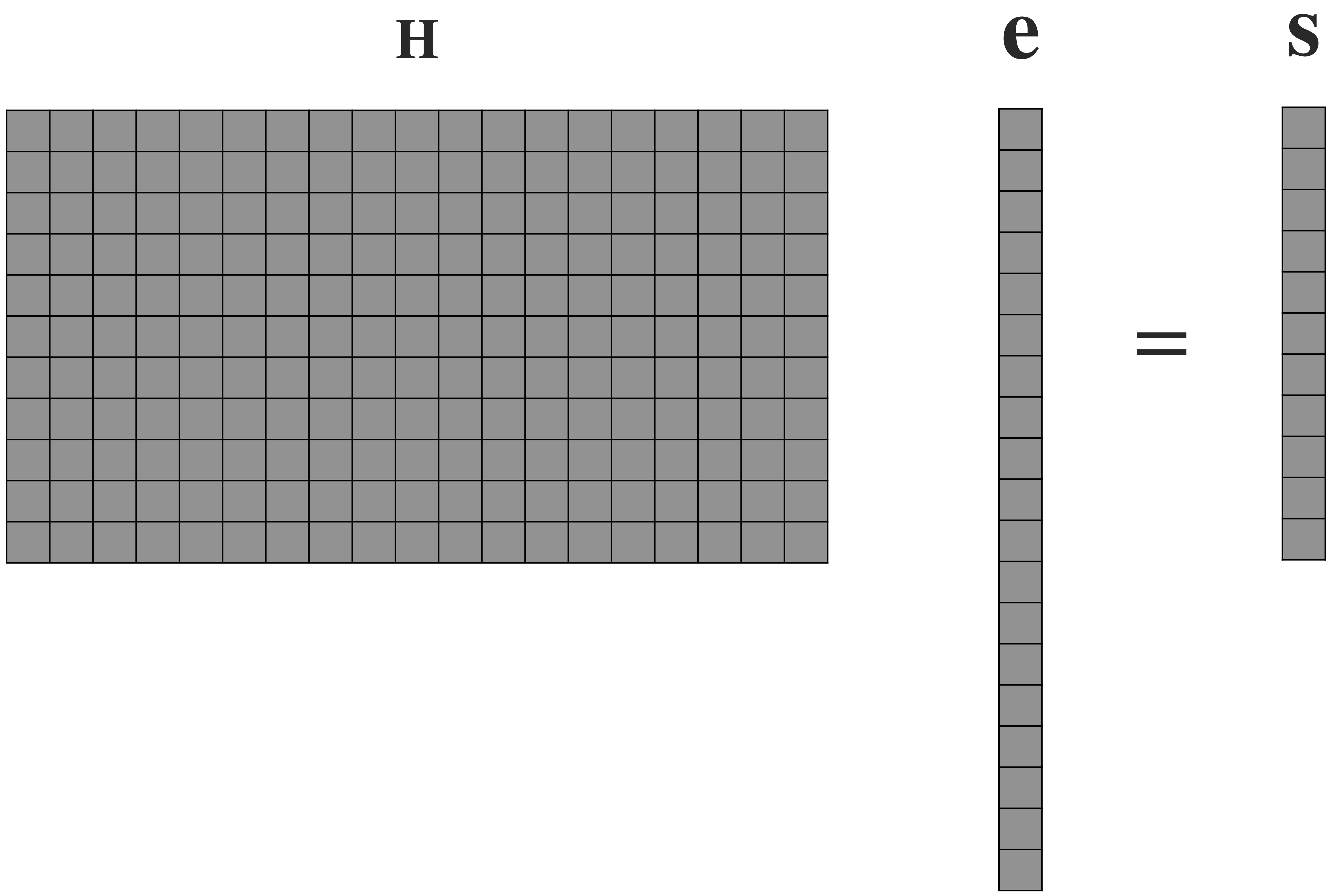

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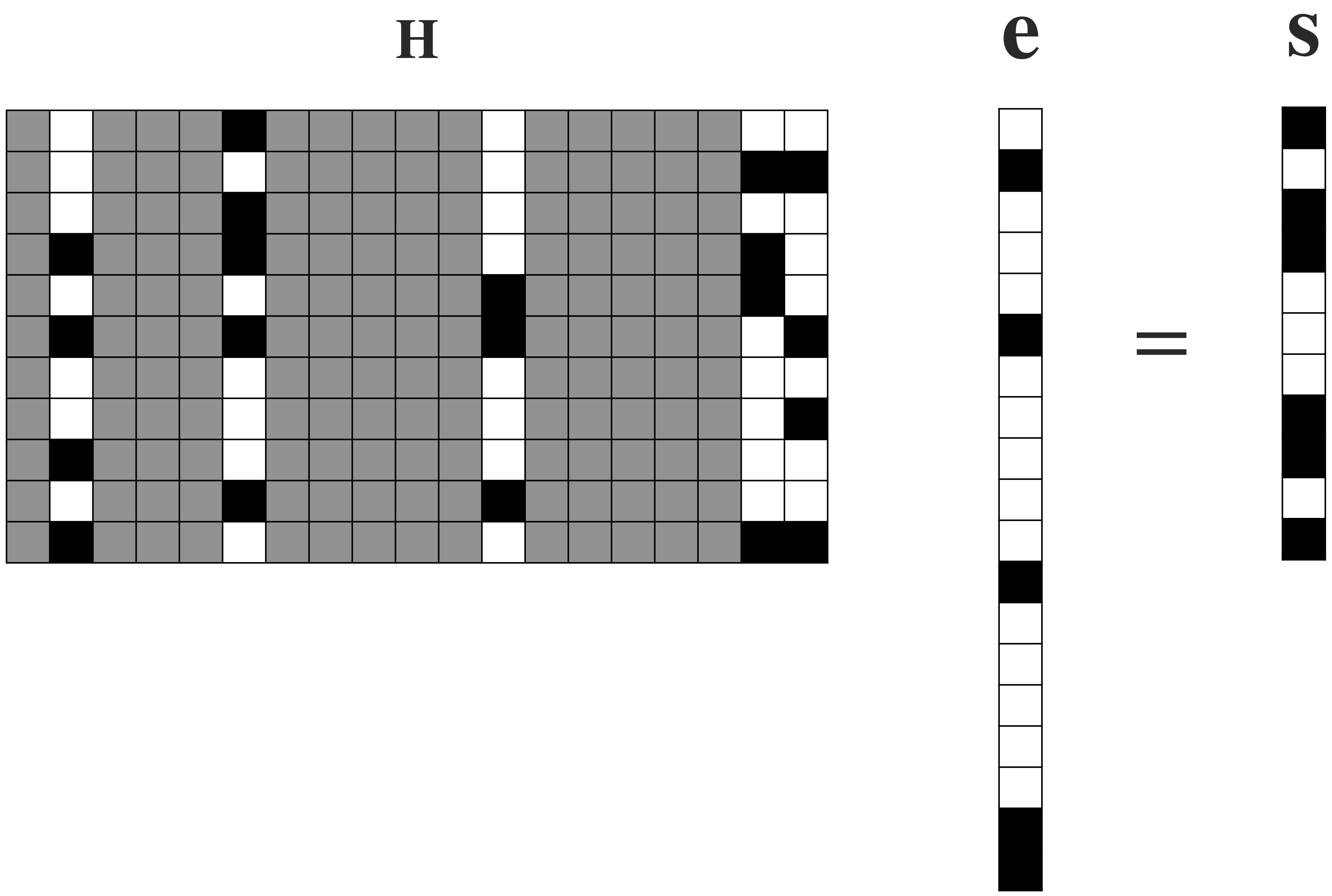

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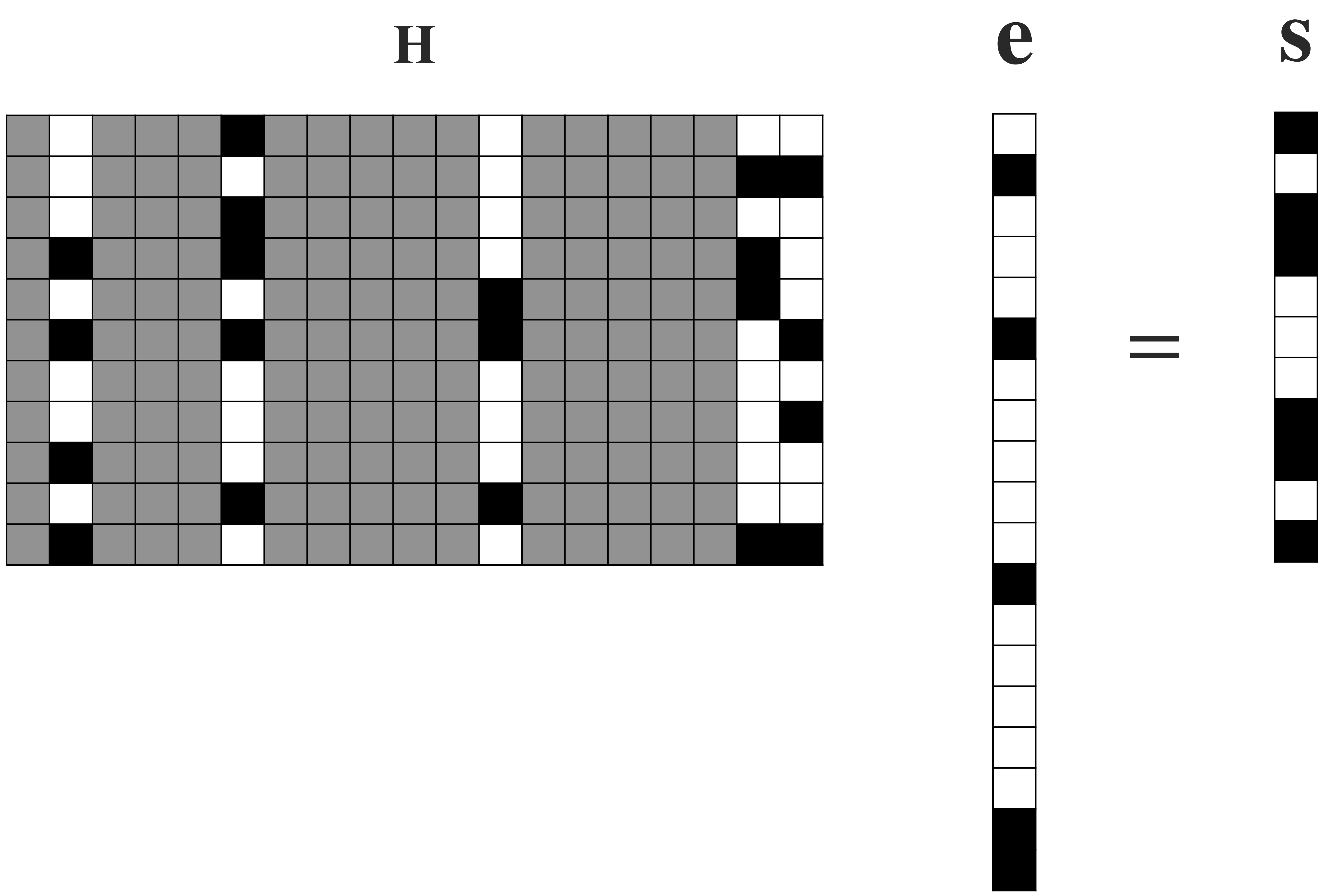

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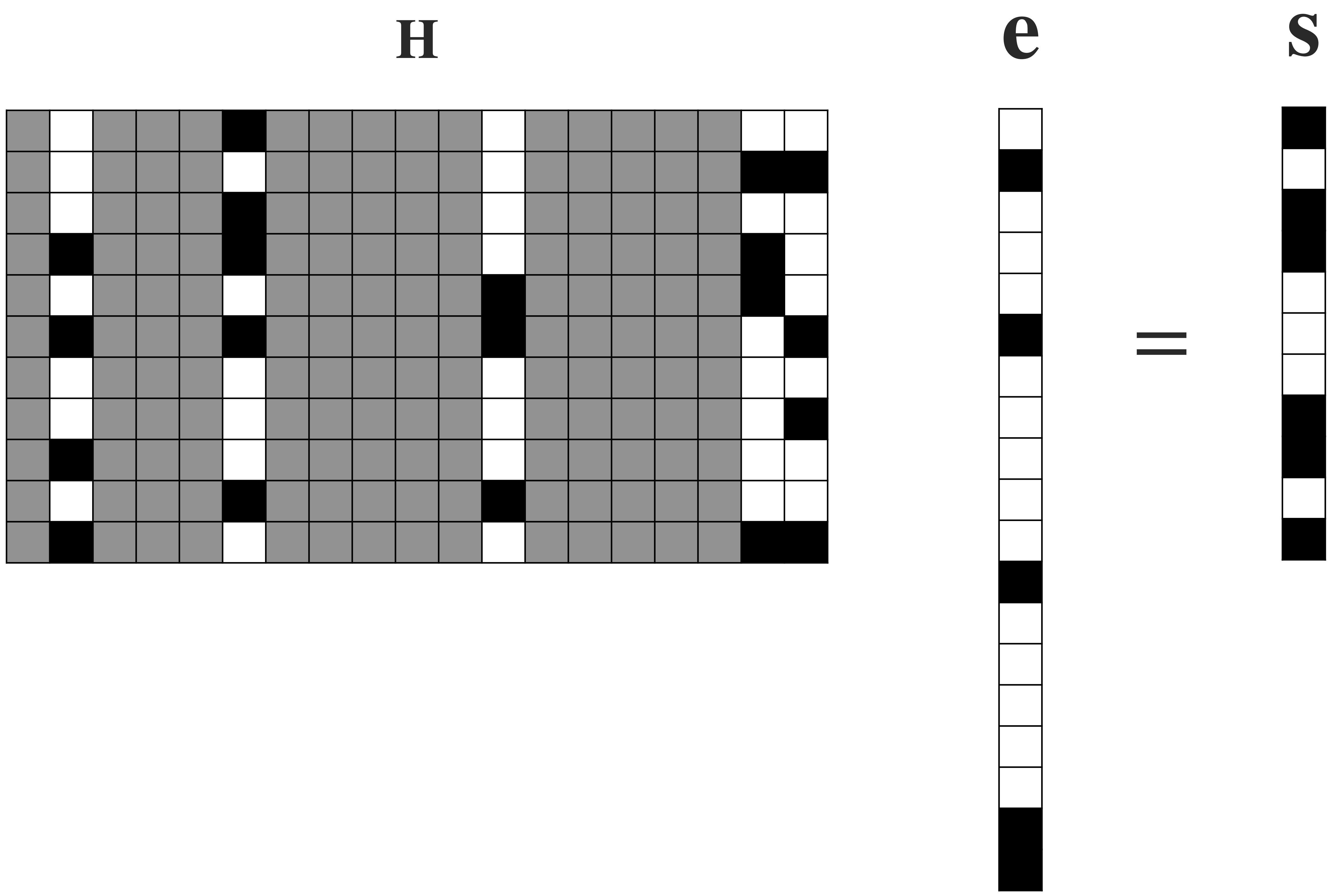
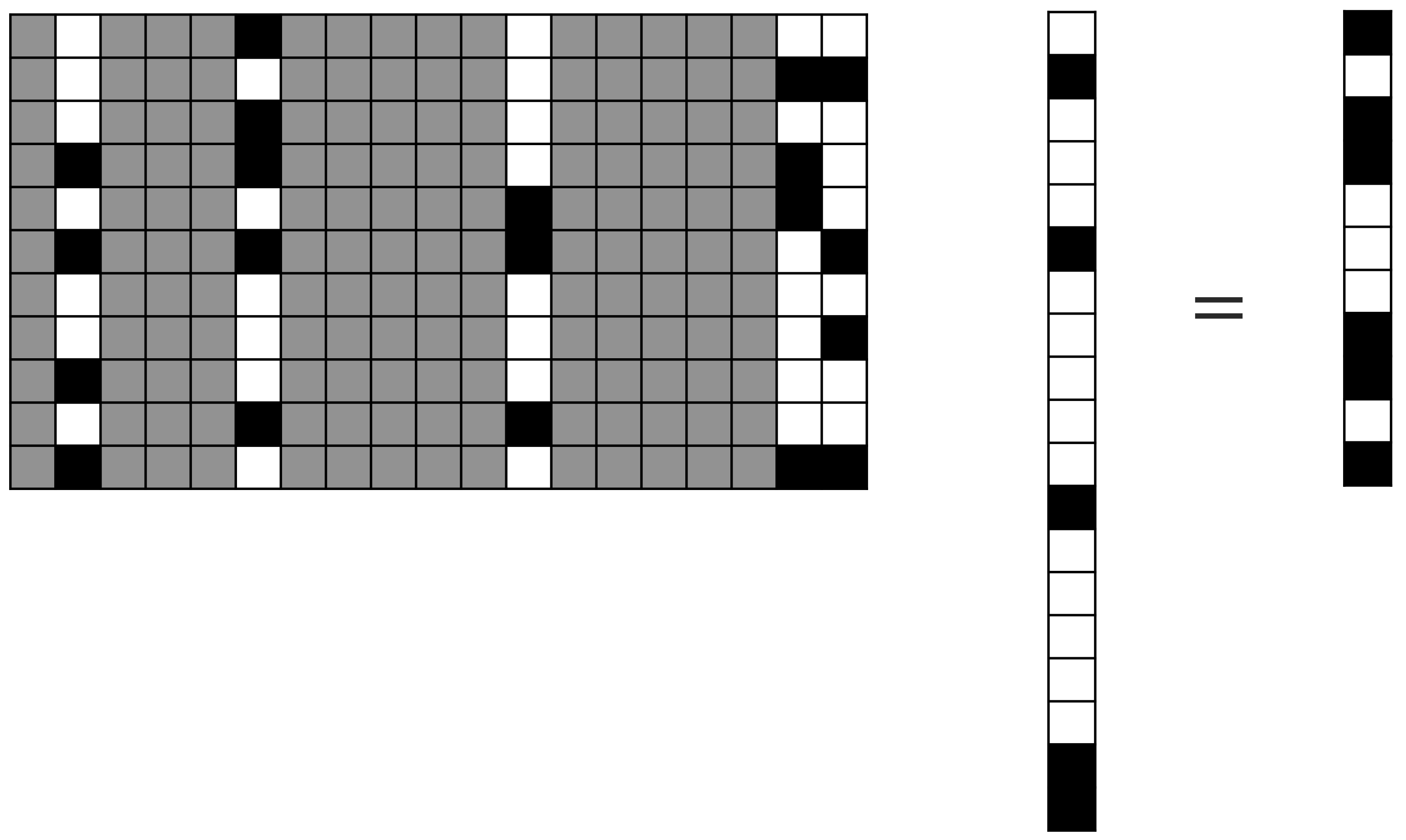


$\mathbf{s}$ is equal to the sum of the columns where e_i is nonzero.

Pick any group of t columns of $\mathbf{H}$, add them and compare with $\mathbf{s}$.

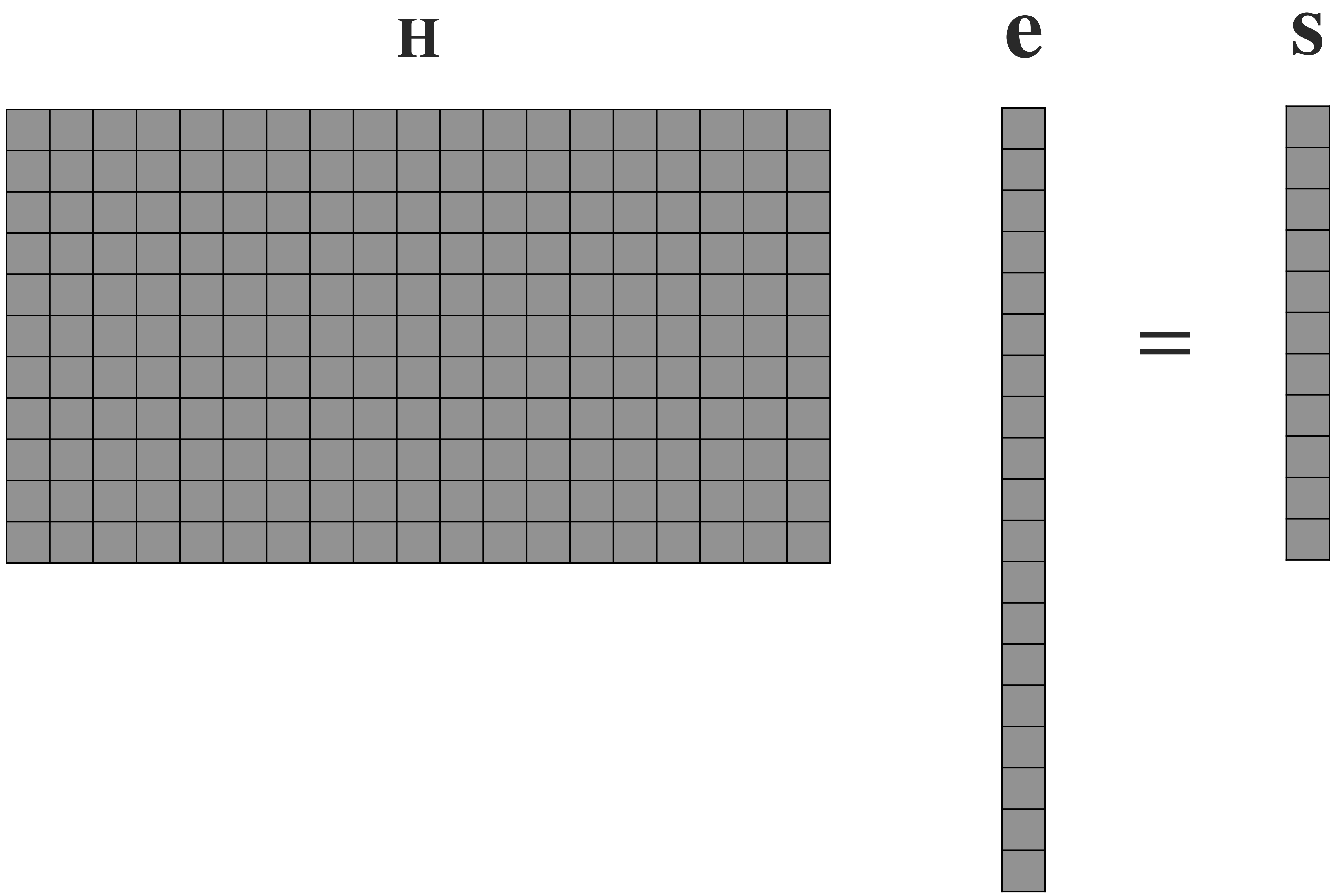
Brute force: complexity



Pick any group of t columns of $\mathbf{H}$, add them and compare with $\mathbf{s}$.

Cost: $\binom{n}{t}$ sums of t columns.

Prange's attack



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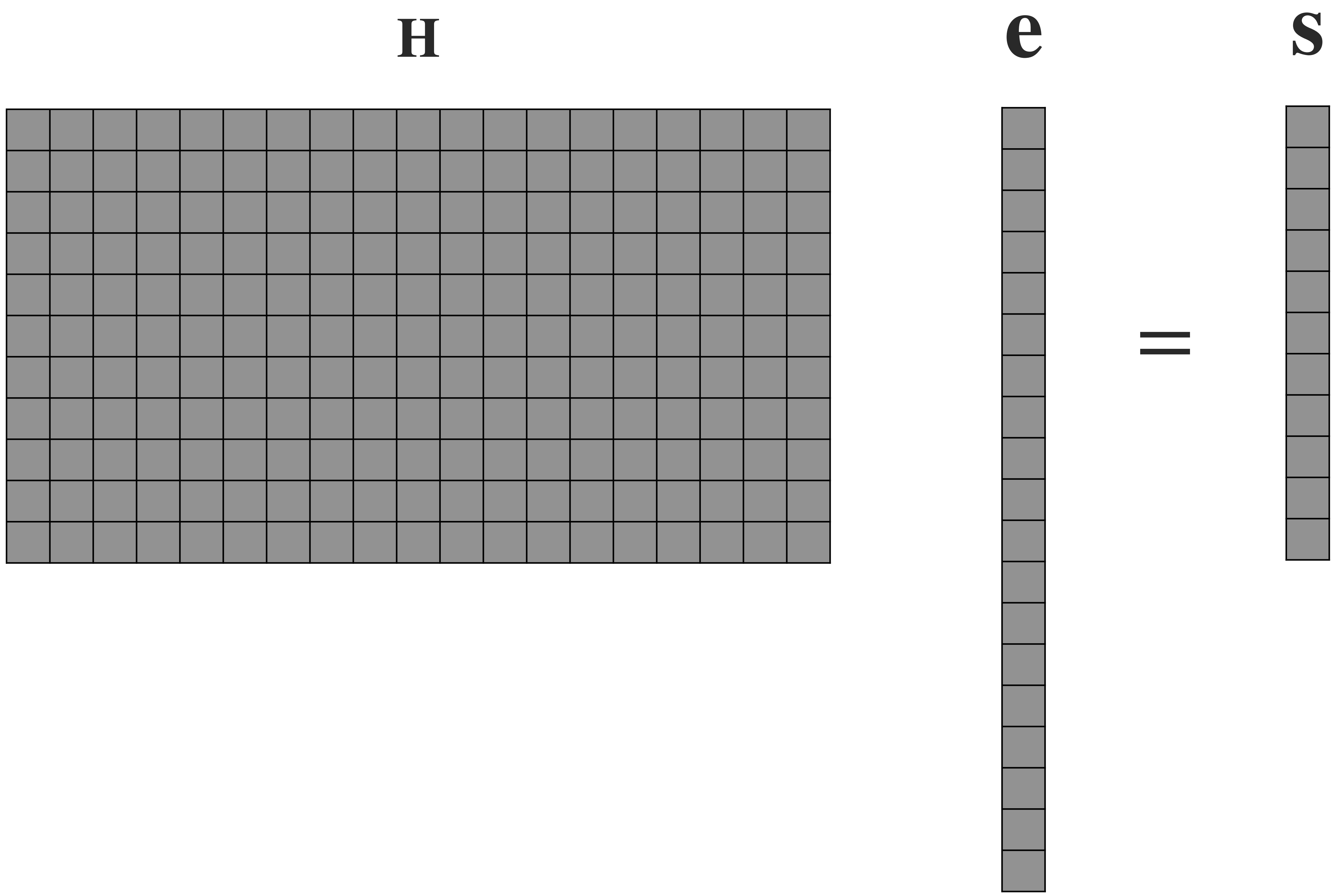
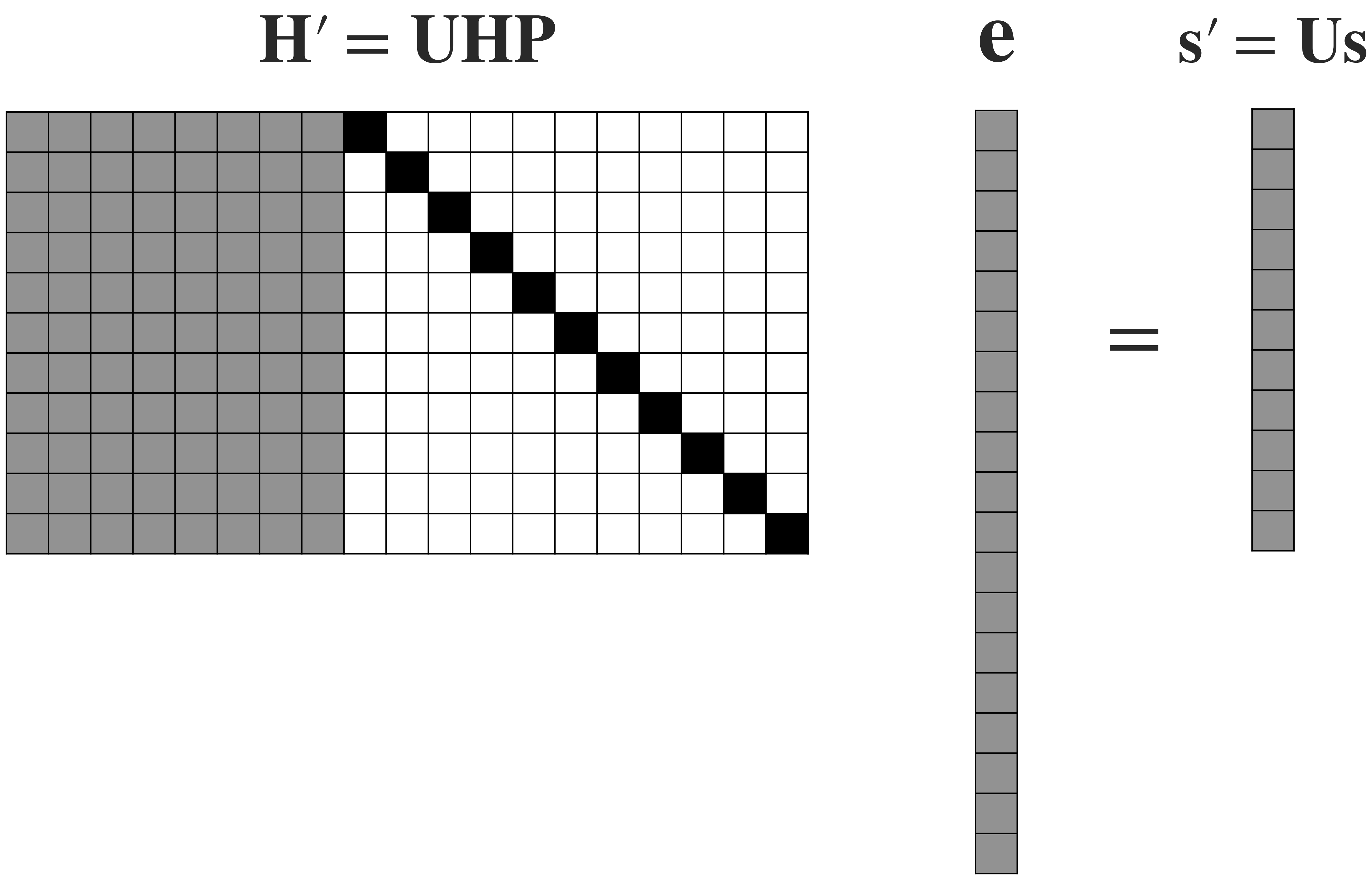


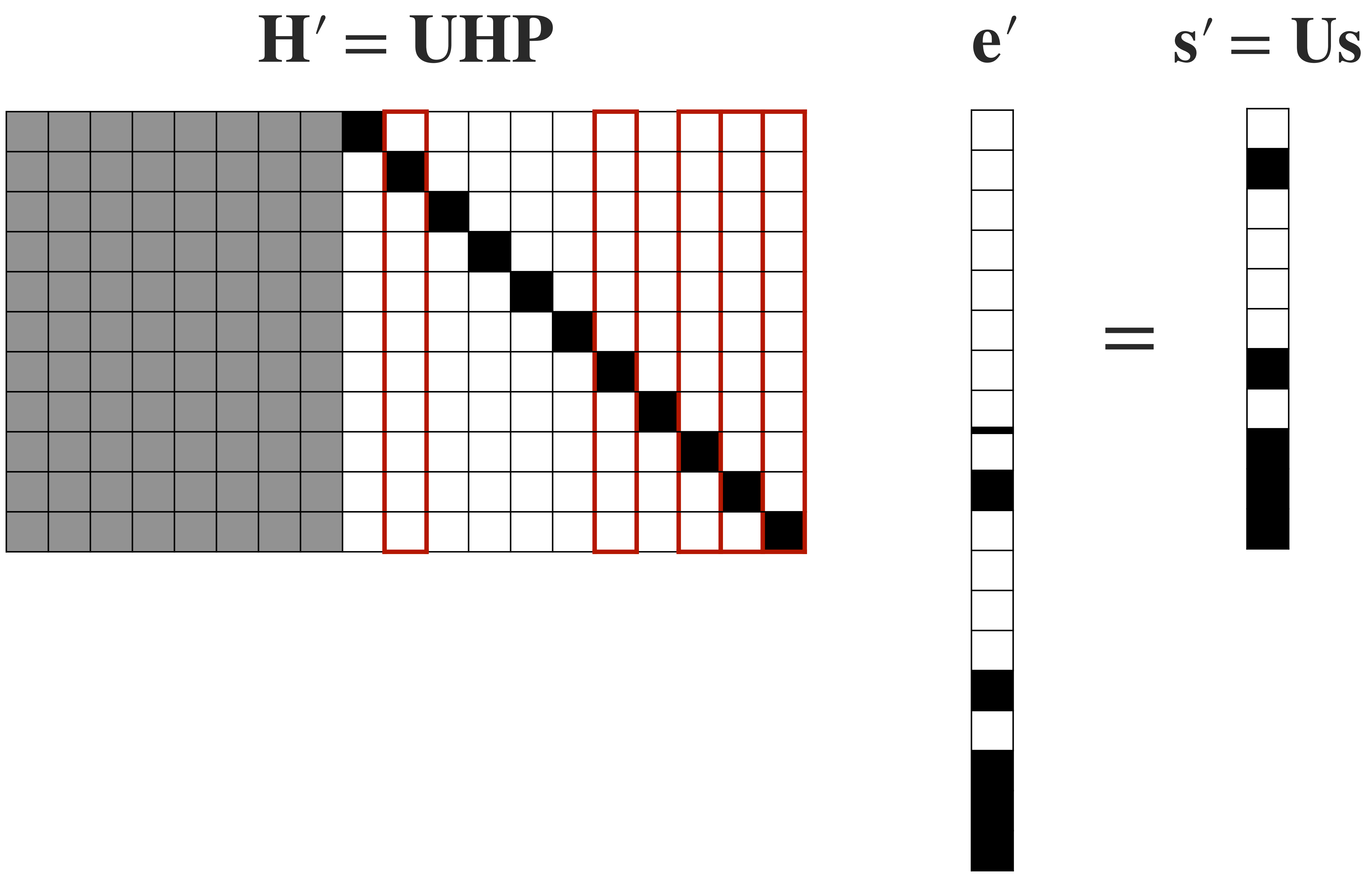
→ Permute **H** and bring to systematic form.

Prange's attack



→ Permute $\mathbf{H}$ and bring to systematic form.

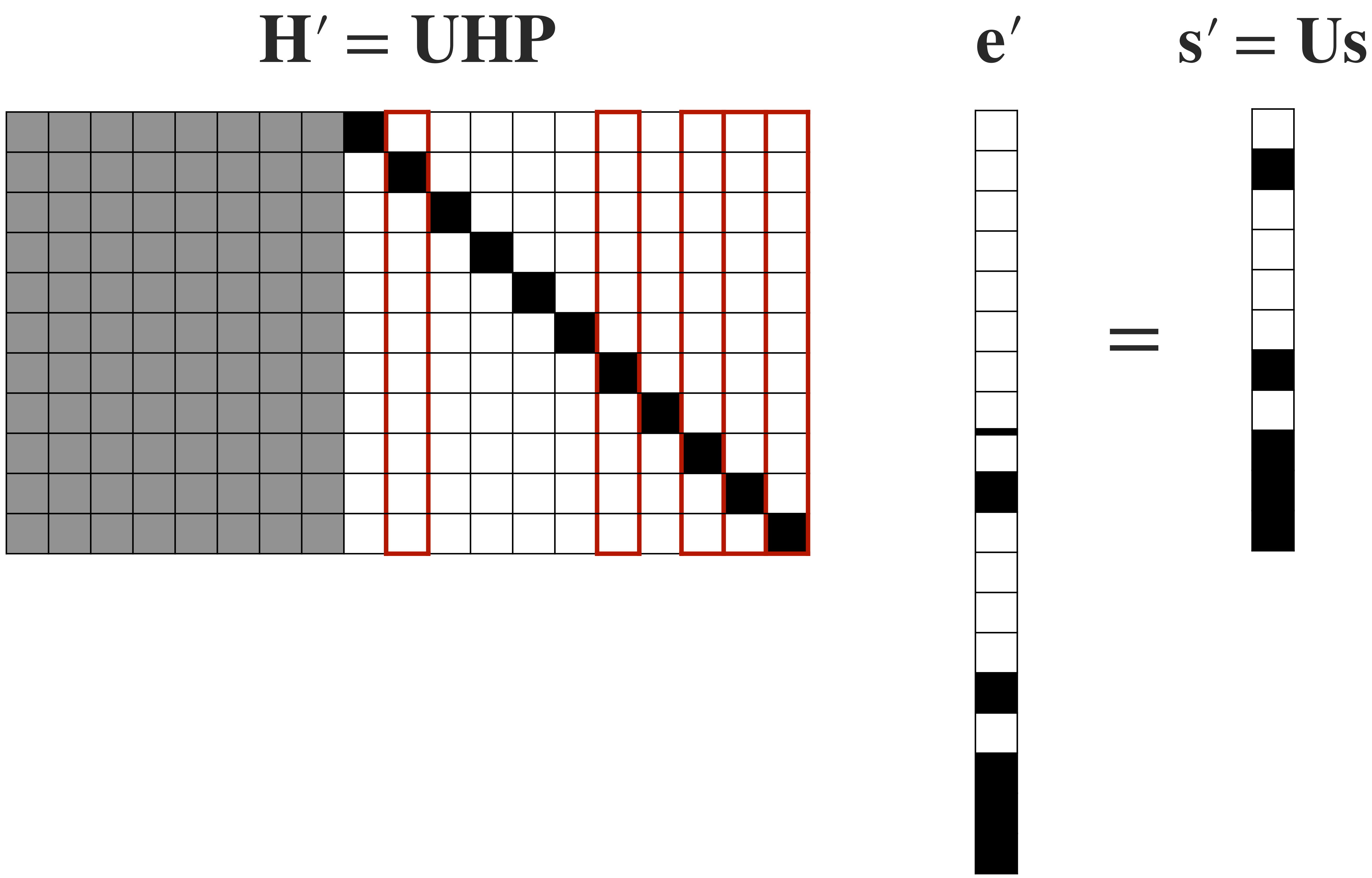
Prange's attack



Permute $\mathbf{H}$ and bring to systematic form.

Suppose that all t errors are in the identity (right) part. Then $\mathbf{e}' = (000\ldots)||\mathbf{Us}$ and $\mathrm{wt}(\mathbf{Us}) = t$.

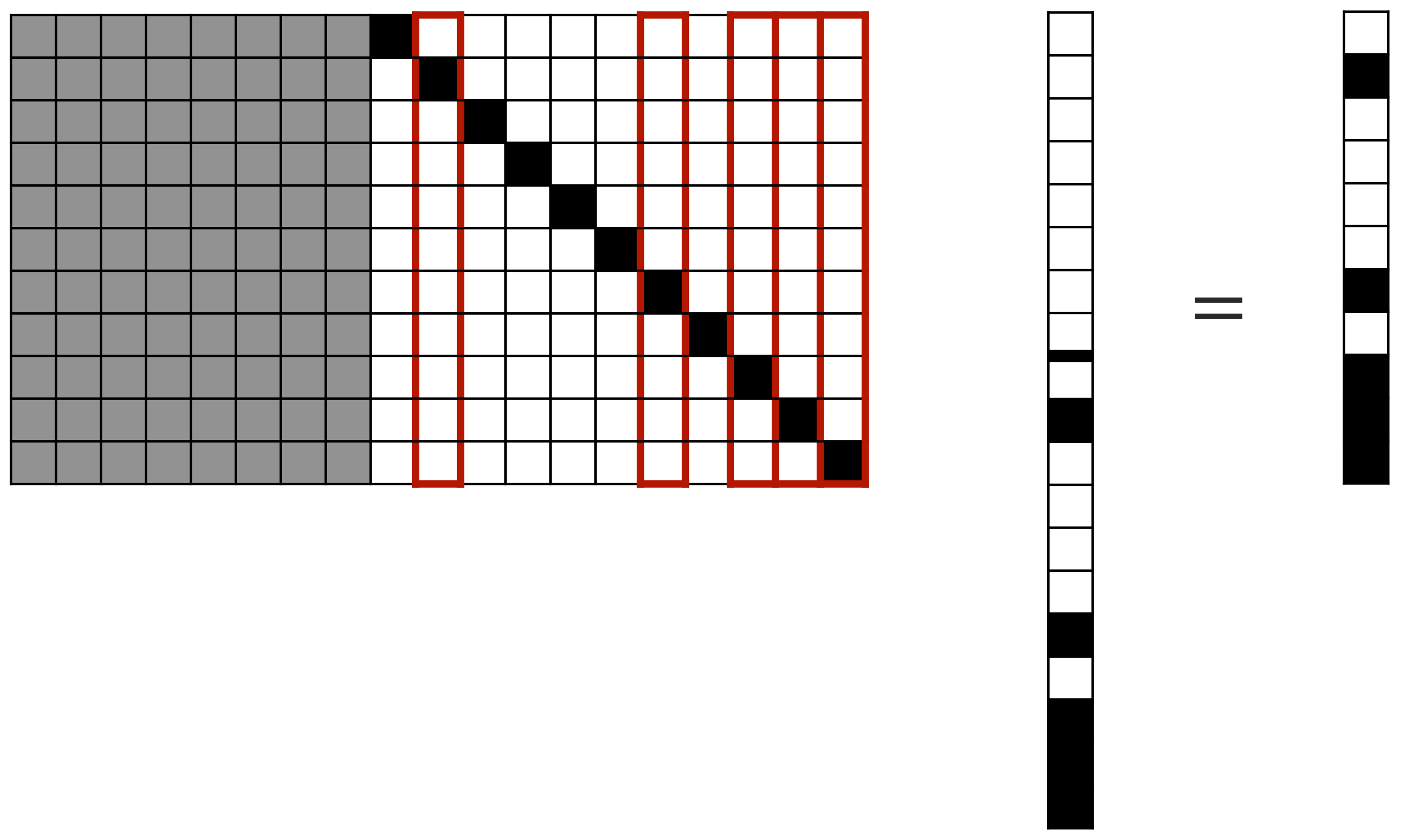
Prange's attack



- Permute $\mathbf{H}$ and bring to systematic form.

  Suppose that all t errors are in the identity (right) part. Then $\mathbf{e}' = (000\ldots)||\mathbf{Us}$ and $\mathrm{wt}(\mathbf{Us}) = t$.
- If $\mathrm{wt}(\mathbf{Us}) = t$, then output unpermuted version of $\mathbf{e}'$.
- Else, return to the first step and rerandomize: choose a new permutation.

Prange’s attack: complexity



- Permute $\mathbf{H}$ and bring to systematic form.
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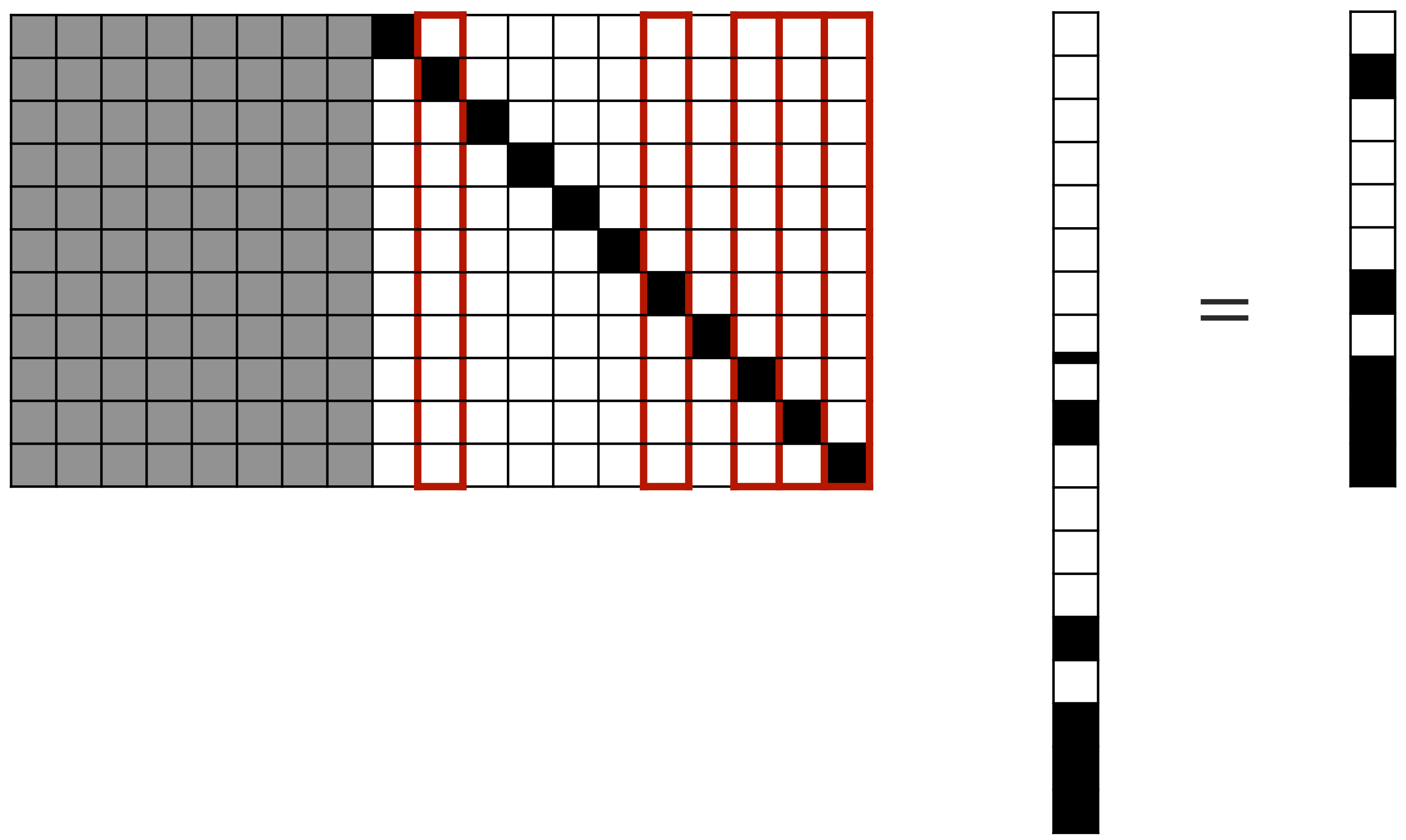
Prange's attack: complexity

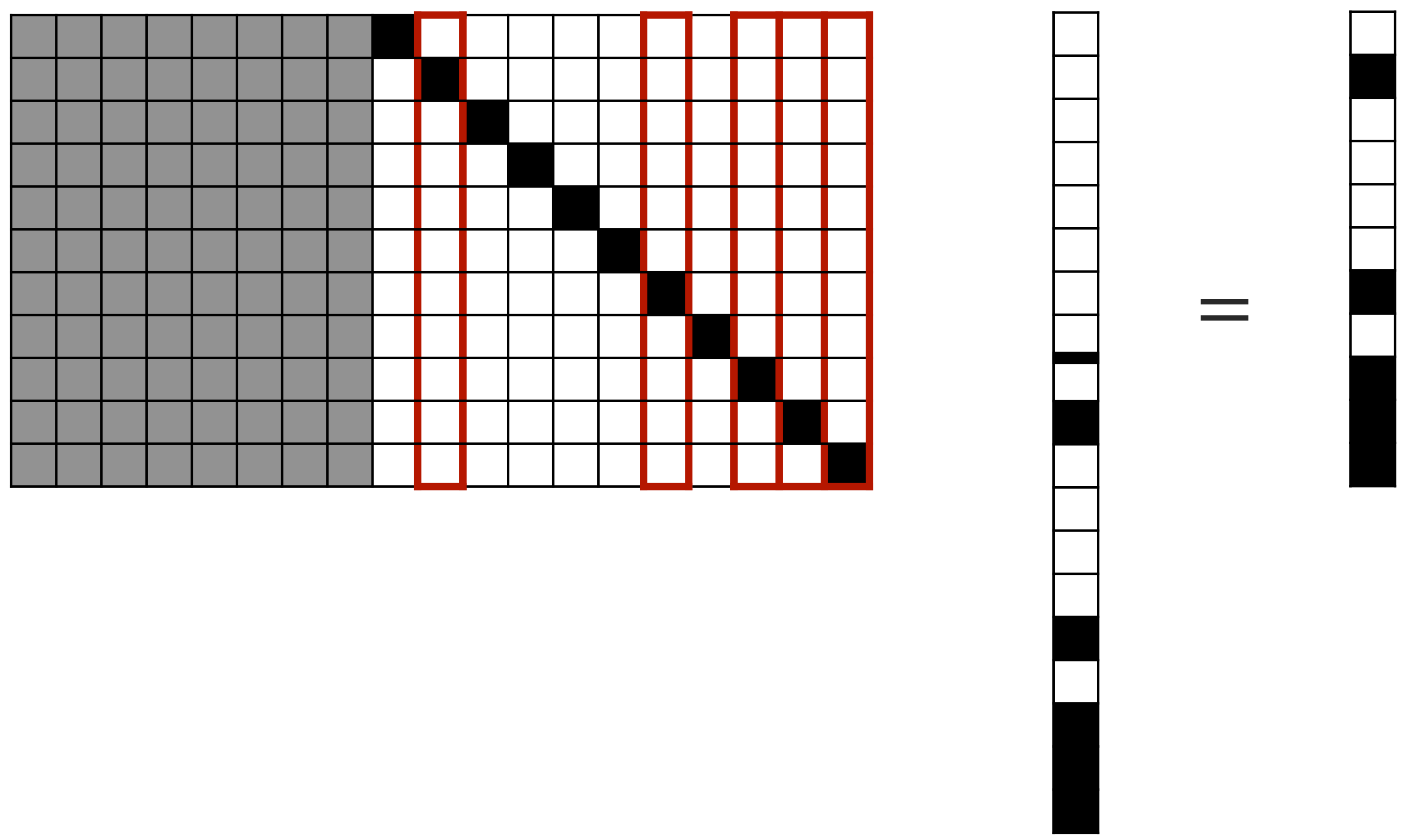


- Permute $\mathbf{H}$ and bring to systematic form.
- If $\mathrm{wt}(\mathbf{U}\mathbf{s}) = t$, then output unpermuted version of $\mathbf{e}$.
- Else, return to the first step and rerandomize: choose a new permutation.

All errors are in the identity part

Probability that we are in the correct configuration: $\dfrac{\binom{n-k}{t}}{\binom{n}{t}}$.

Prange's attack: complexity



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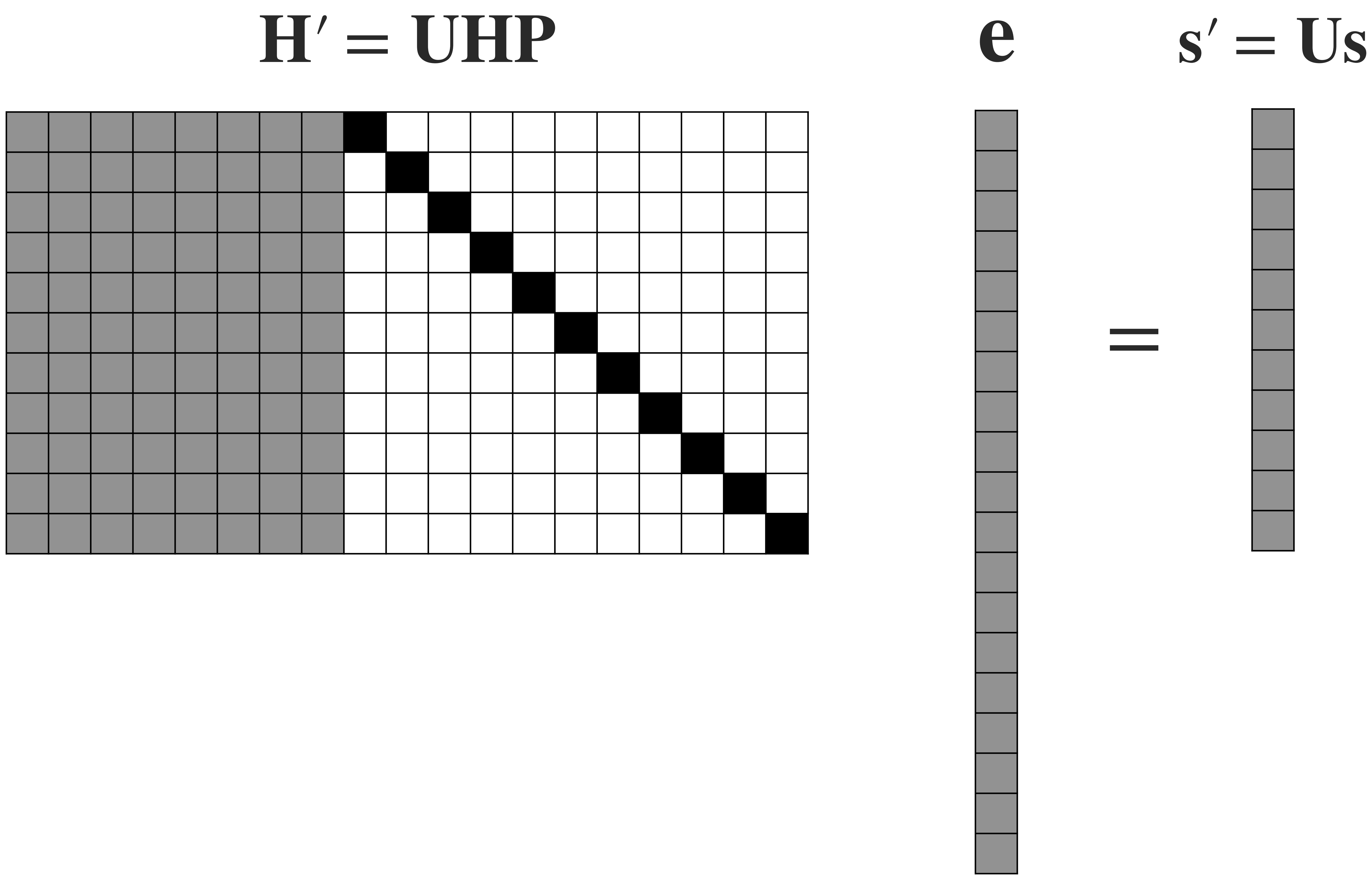
All errors are in the identity part

Probability that we are in the correct configuration: $\dfrac{\binom{n-k}{t}}{\binom{n}{t}}$.

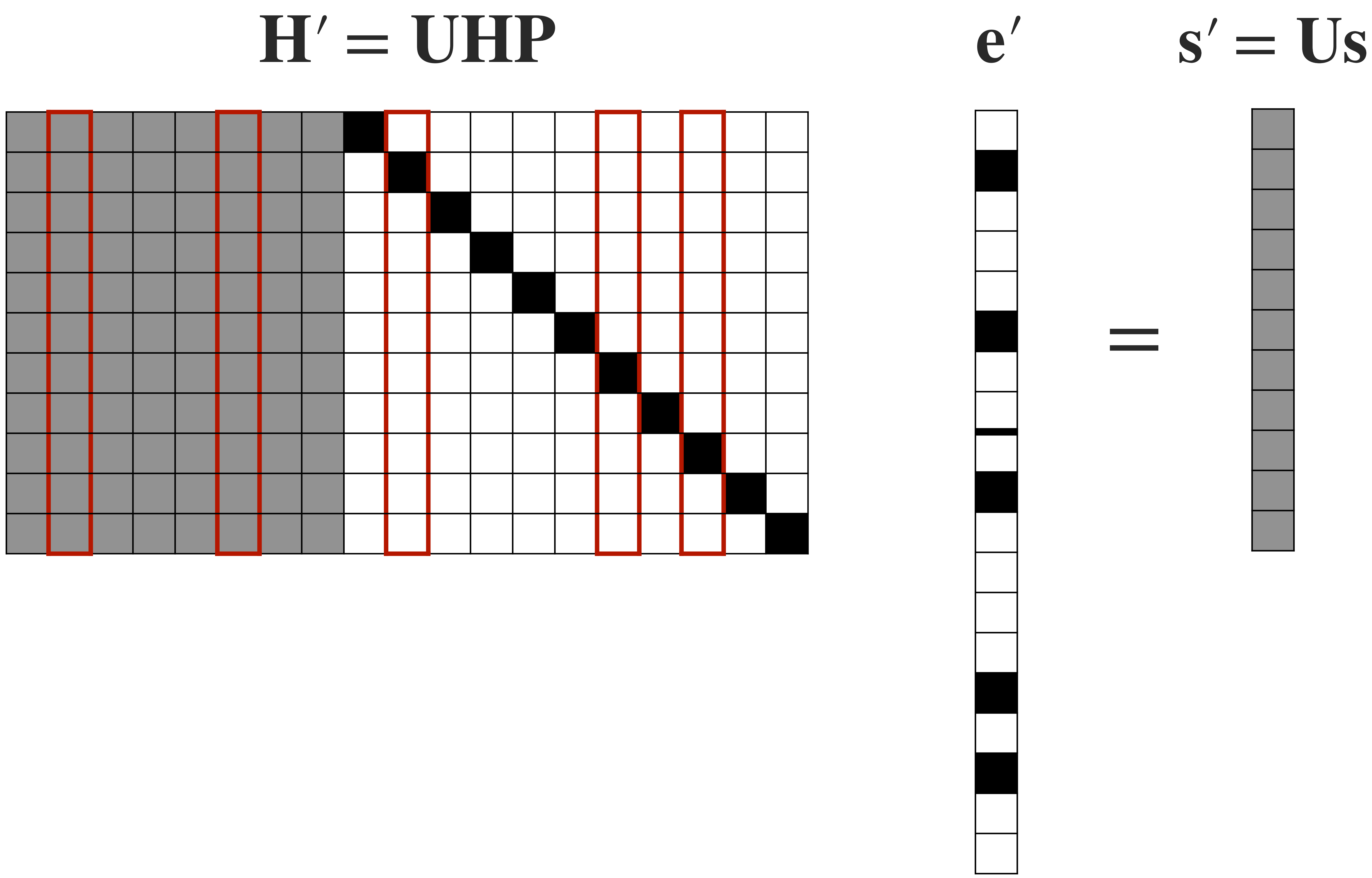
Cost: $\dfrac{\binom{n}{t}}{\binom{n-k}{t}}$ matrix operations.

Lee-Brickell attack



→ Permute **H** and bring to systematic form.

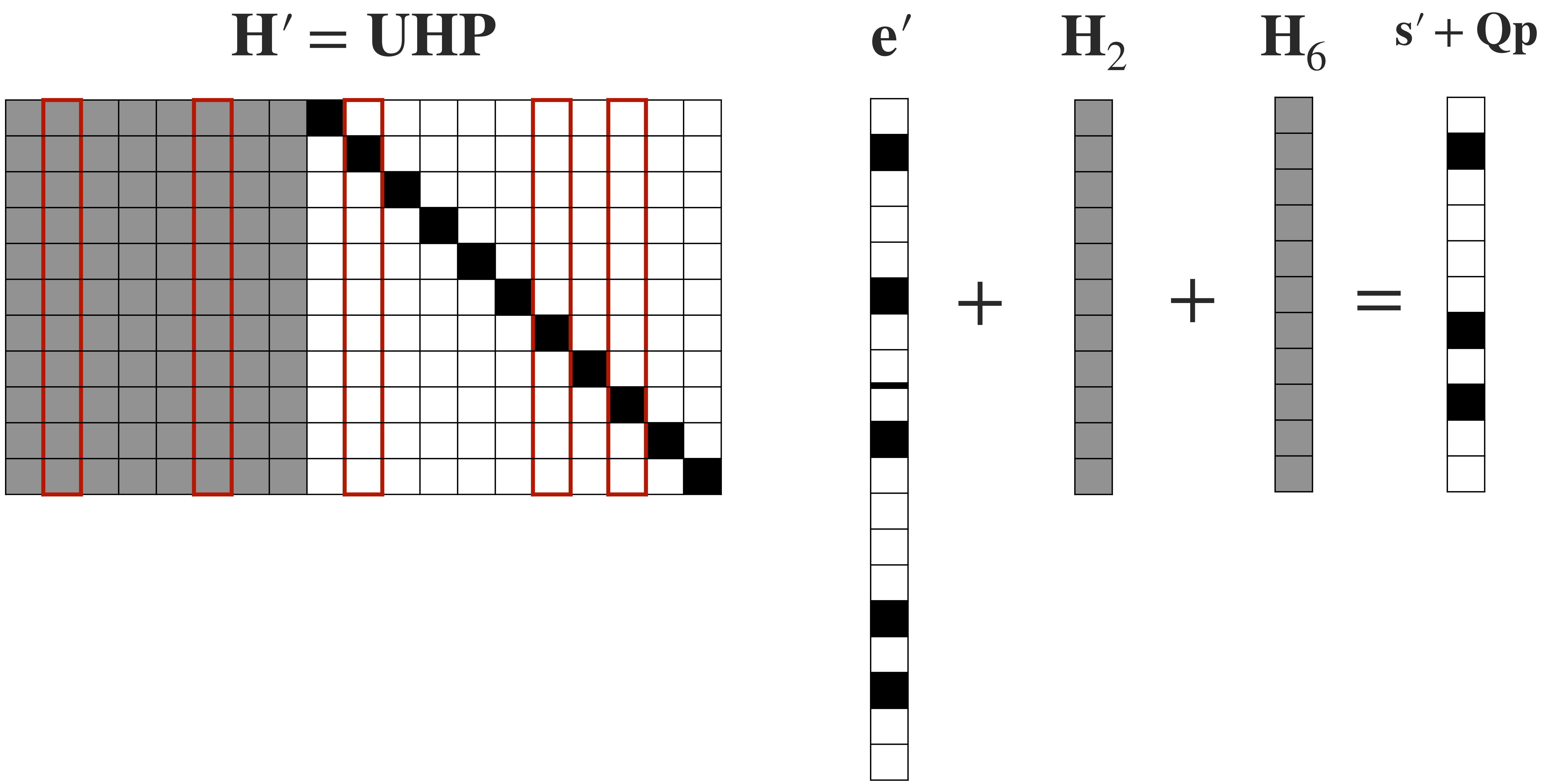
Lee-Brickell attack



Permute $\mathbf{H}$ and bring to systematic form.

Suppose that there are $(t - p)$ errors are in the identity (right) part and p errors in the left part.

Lee-Brickell attack

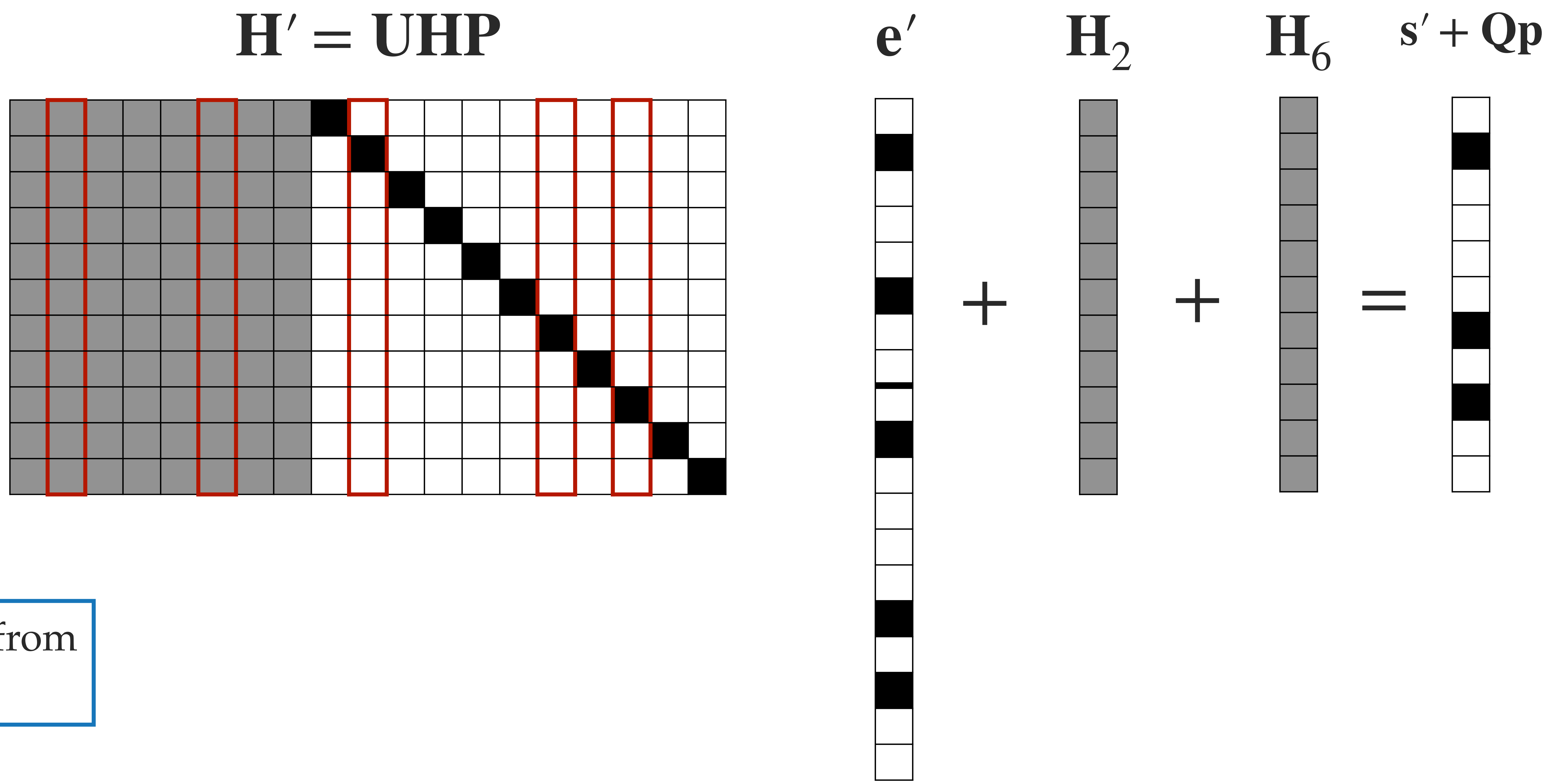


Permute $\mathbf{H}$ and bring to systematic form.

Suppose that there are $(t - p)$ errors are in the identity (right) part and p errors in the left part.

Then, $\mathbf{s}'$ is random-looking, but $\mathbf{s}'$ summed with the error columns on the left has weight $t - p$: $\mathrm{wt}(\mathbf{s}' + \mathbf{Qp}) = t - p$.

Lee-Brickell attack



Let $\mathbf{p}$ be a vector chosen from $\{\mathbf{p} \in \mathbb{F}^k \mid \mathrm{wt}(\mathbf{p}) = p\}$

Permute $\mathbf{H}$ and bring to systematic form.

Suppose that there are $(t - p)$ errors are in the identity (right) part and p errors in the left part.

Then, $\mathbf{s}'$ is random-looking, but $\mathbf{s}'$ summed with the error columns on the left has weight $t - p$:
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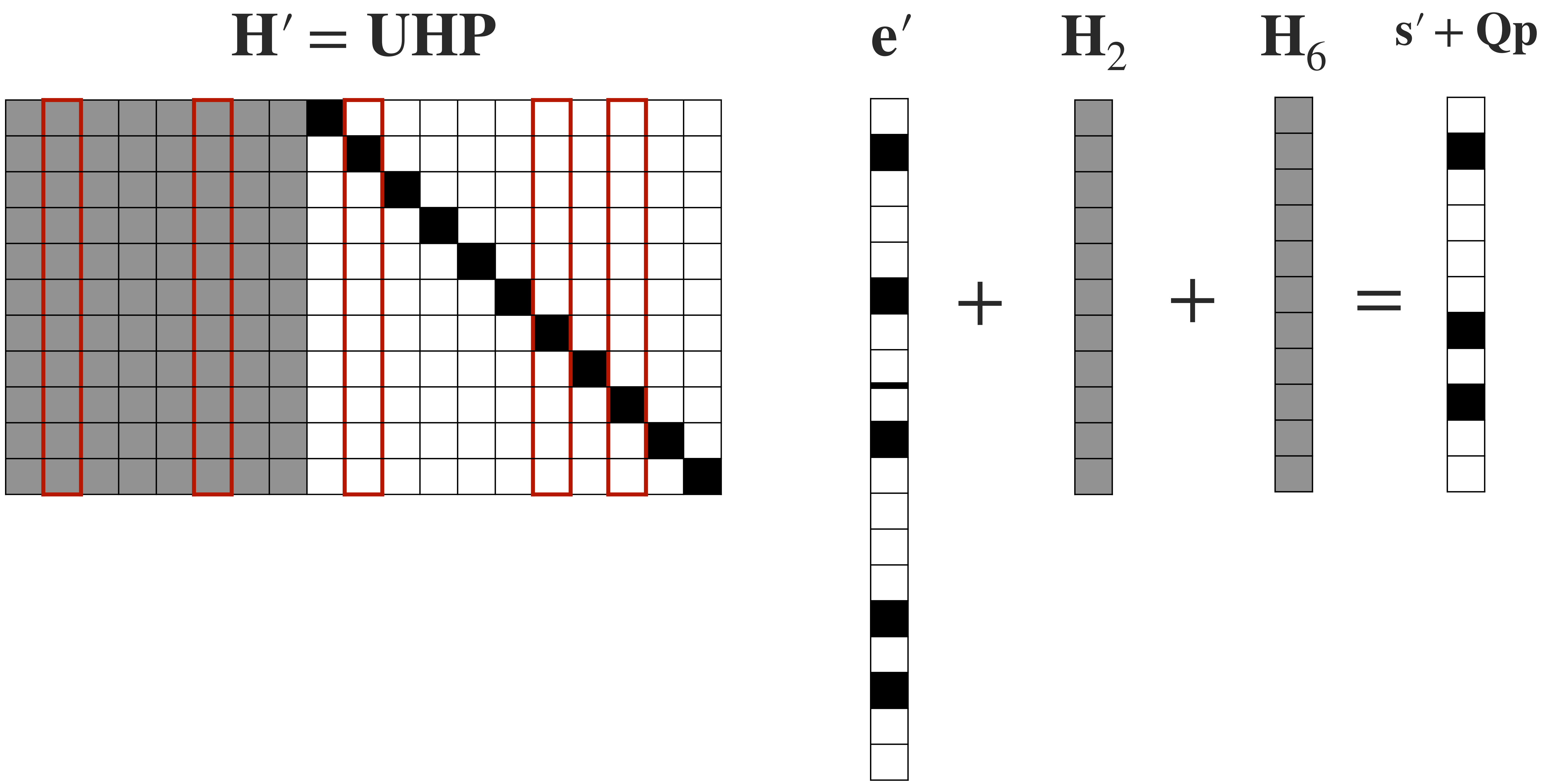
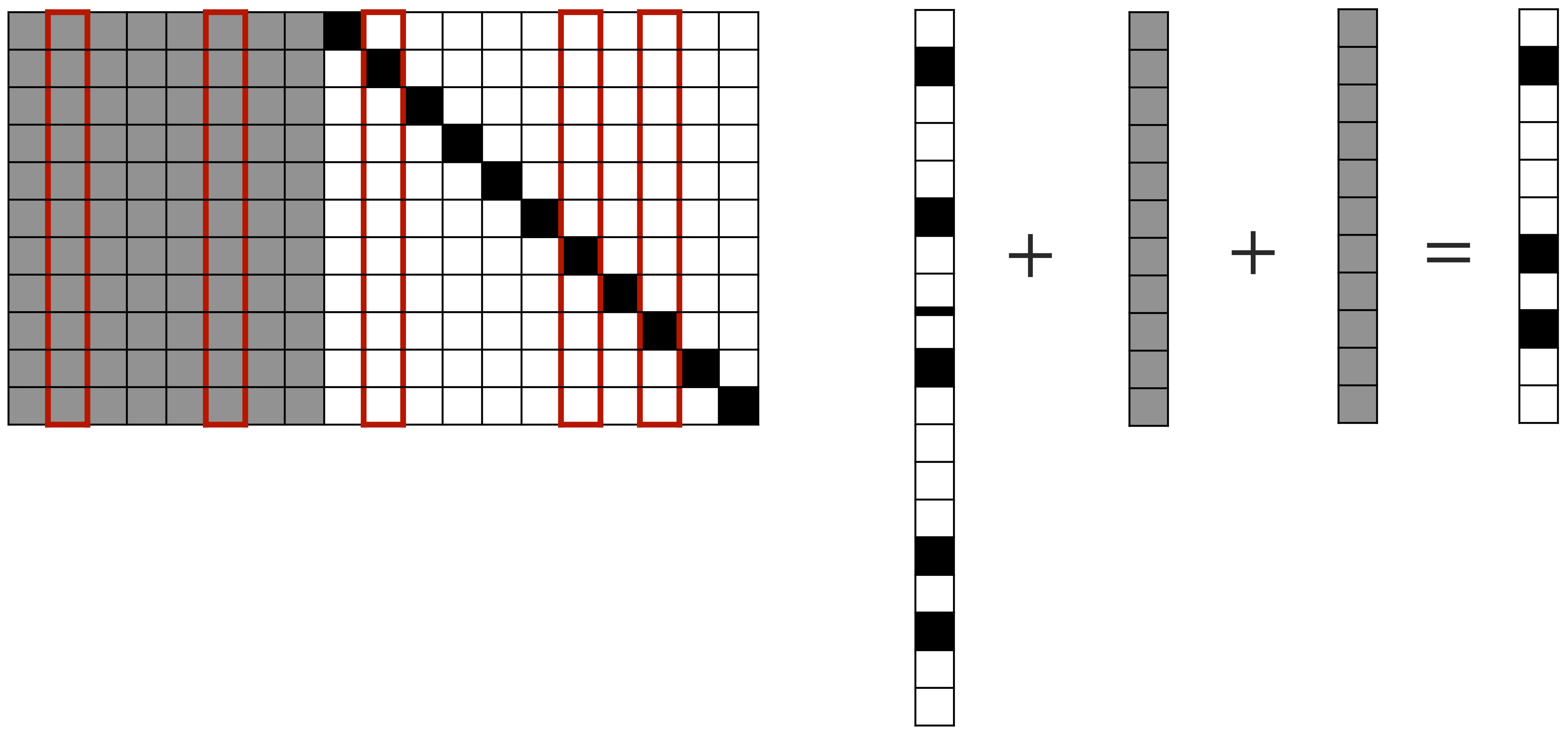
Lee-Brickell attack

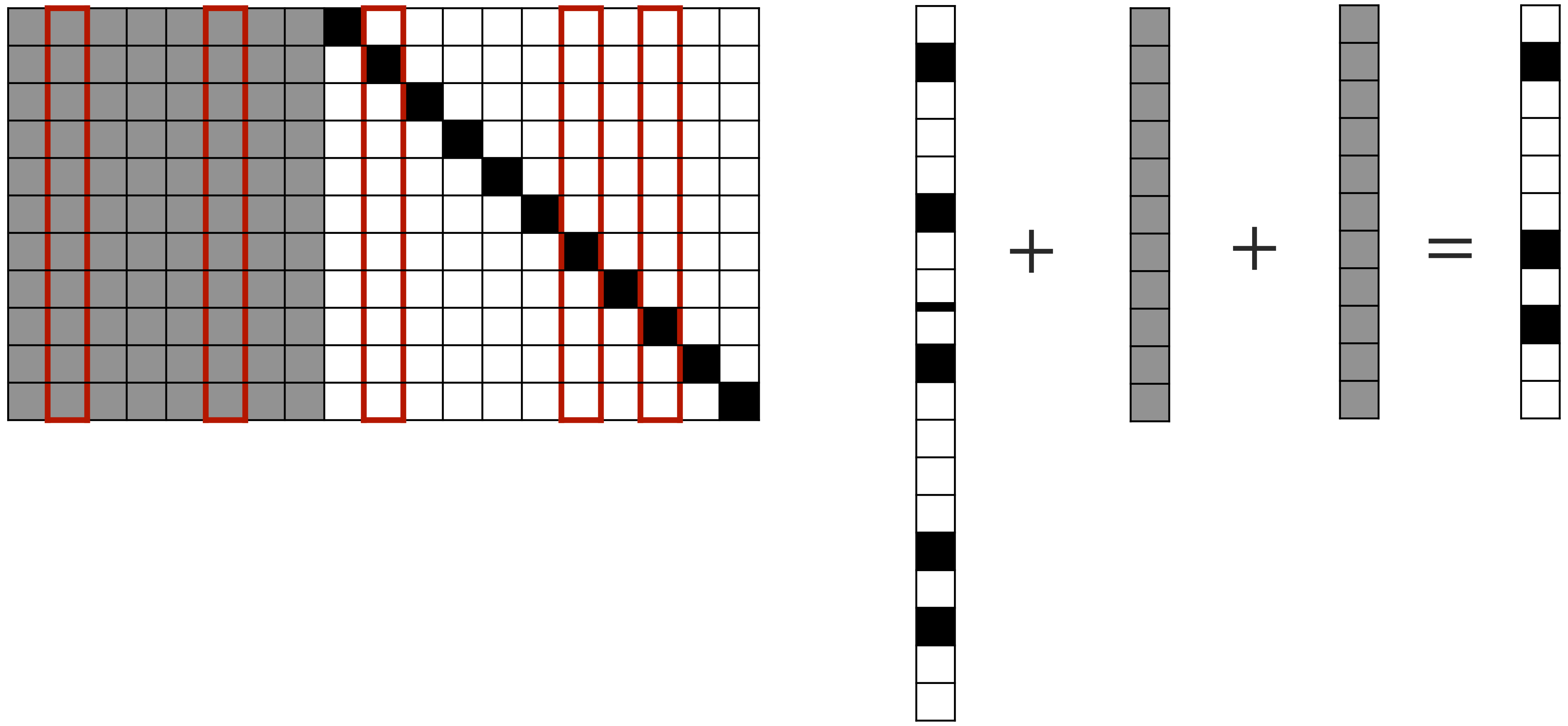


- Permute $\mathbf{H}$ and bring to systematic form.
- Pick p of the columns on the left and compute their sum: $\mathbf{Qp}$.
- If $\mathrm{wt}(\mathbf{s}' + \mathbf{Qp}) = t - p$ then put $\mathbf{e}' = \mathbf{p} \,||\, (\mathbf{s}' + \mathbf{Qp})$. Output unpermuted version of $\mathbf{e}$.
- Else, return to the second step to choose another subset of columns from $\mathbf{Q}$, or return to the first step and rerandomize.

Lee-Brickell attack: complexity



Lee-Brickell attack: complexity

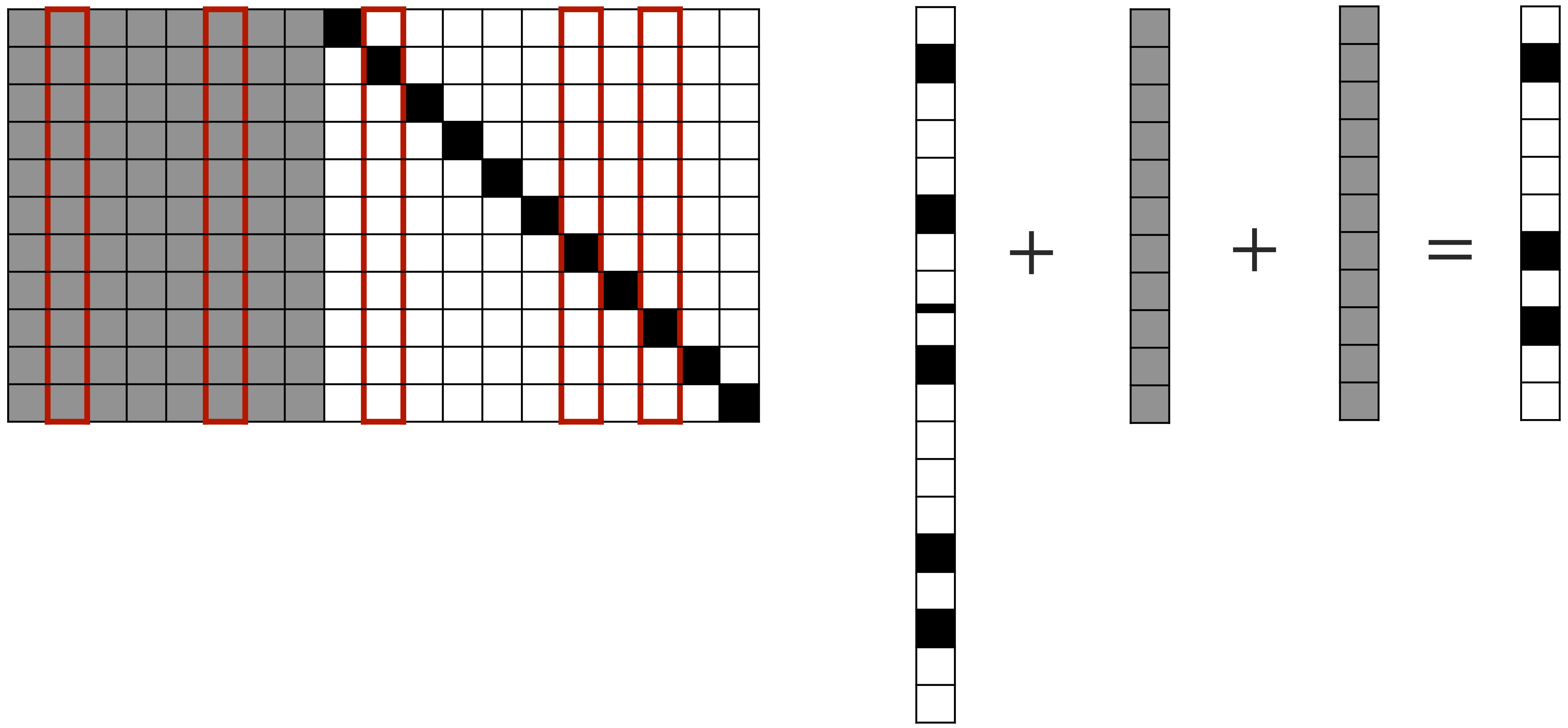


$t - p$ errors are in the identity part

p errors are in the left part

Probability that we are in the correct configuration: $$\frac{\binom{n-k}{t-p}\binom{k}{p}}{\binom{n}{t}}.$$

Lee-Brickell attack: complexity



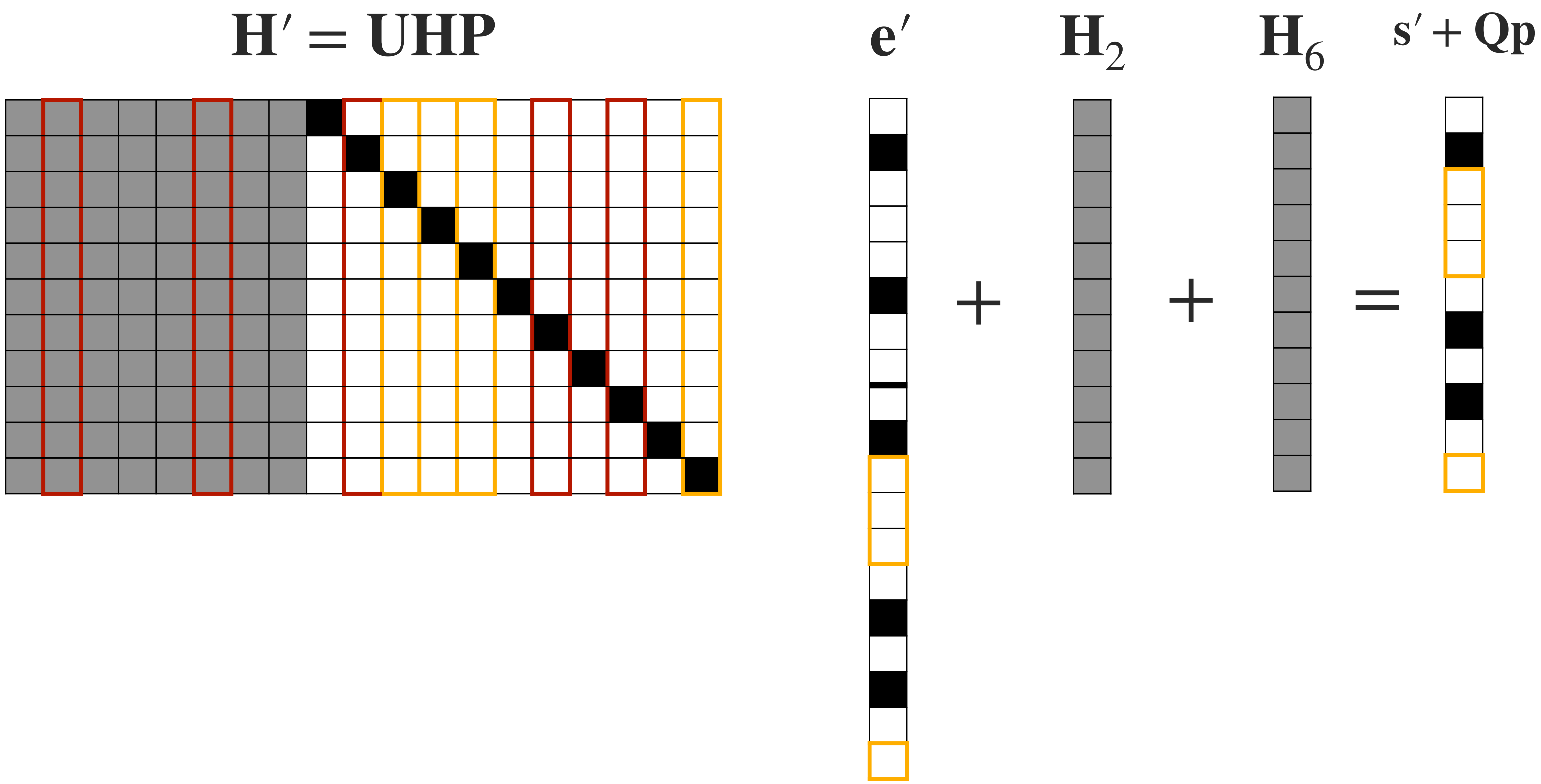
$t-p$ errors are in the identity part

p errors are in the left part

Probability that we are in the correct configuration: $\dfrac{\binom{n-k}{t-p}\binom{k}{p}}{\binom{n}{t}}$.

Cost: $\dfrac{\binom{n}{t}}{\binom{n-k}{t-p}\binom{k}{p}}$ matrix operations $+\binom{k}{p}$ column additions.

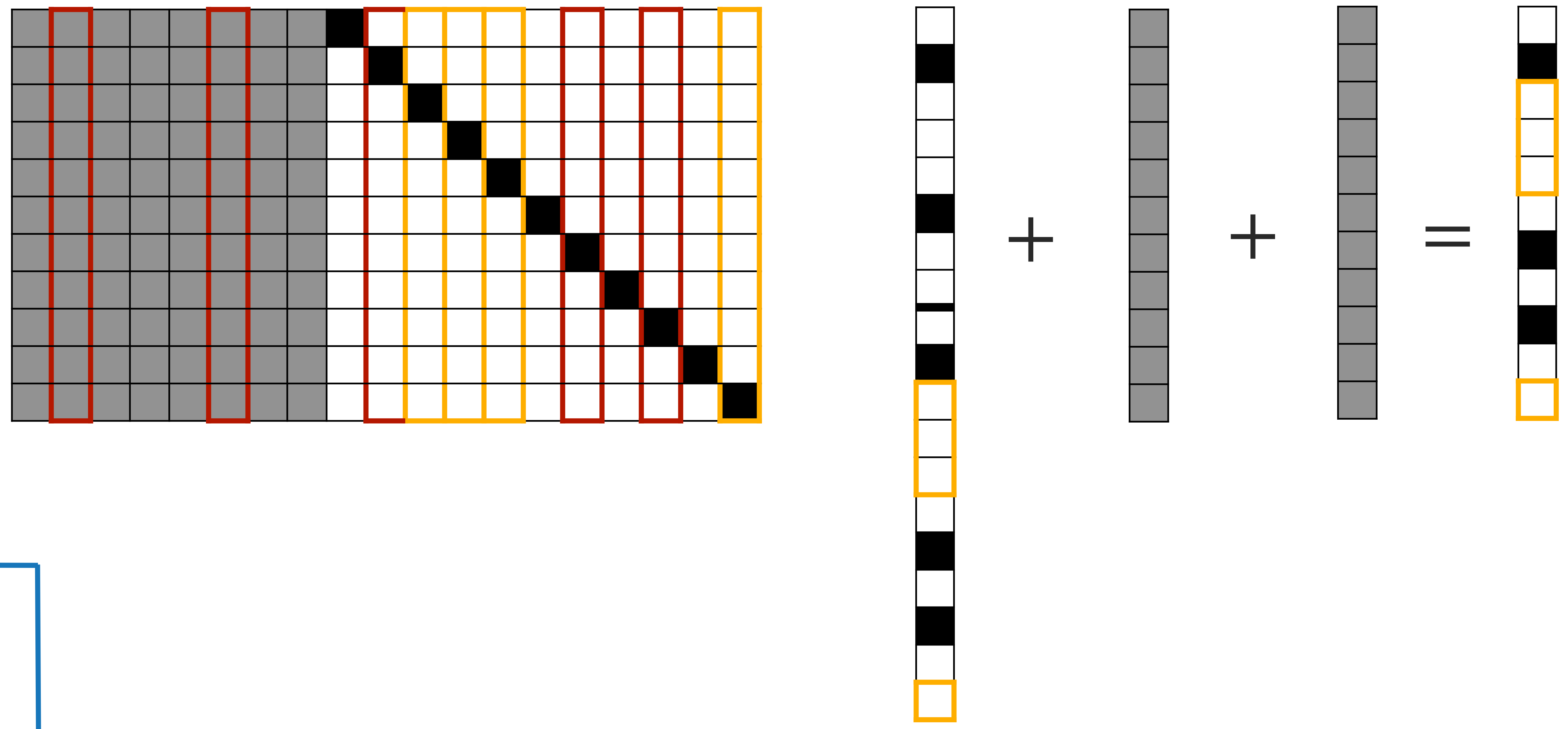
Leon’s attack



Since $\mathbf{s}' + \mathbf{Qp}$ should be of low weight, we check instead if an arbitrary subset of l rows are all zero.

Leon's attack

$\mathbf{H}_L$ denotes the matrix consisting of the rows of $\mathbf{H}$ indexed by L



- Pick a subset L of l rows: $\mathbf{H}_L$.
- Permute $\mathbf{H}$ and bring to systematic form (then $\mathbf{H}_L = \begin{pmatrix} \mathbf{Q}_L & \mathbf{I}_L \end{pmatrix}$).
- Pick p of the columns on the left and compute their sum: $\mathbf{Q}_L\mathbf{p}$.
- If $\mathrm{wt}(\mathbf{s}'_L + \mathbf{Q}_L\mathbf{p}) = 0$
  - If $\mathrm{wt}(\mathbf{s}' + \mathbf{Q}\mathbf{p}) = t - p$ then put $\mathbf{e}' = \mathbf{p}\,||\,(\mathbf{s}' + \mathbf{Q}\mathbf{p})$. Output unpermuted version of $\mathbf{e}$.
  - Else, return to the third step to choose another subset of columns from $\mathbf{Q}$, or return to the second step and rerandomize.
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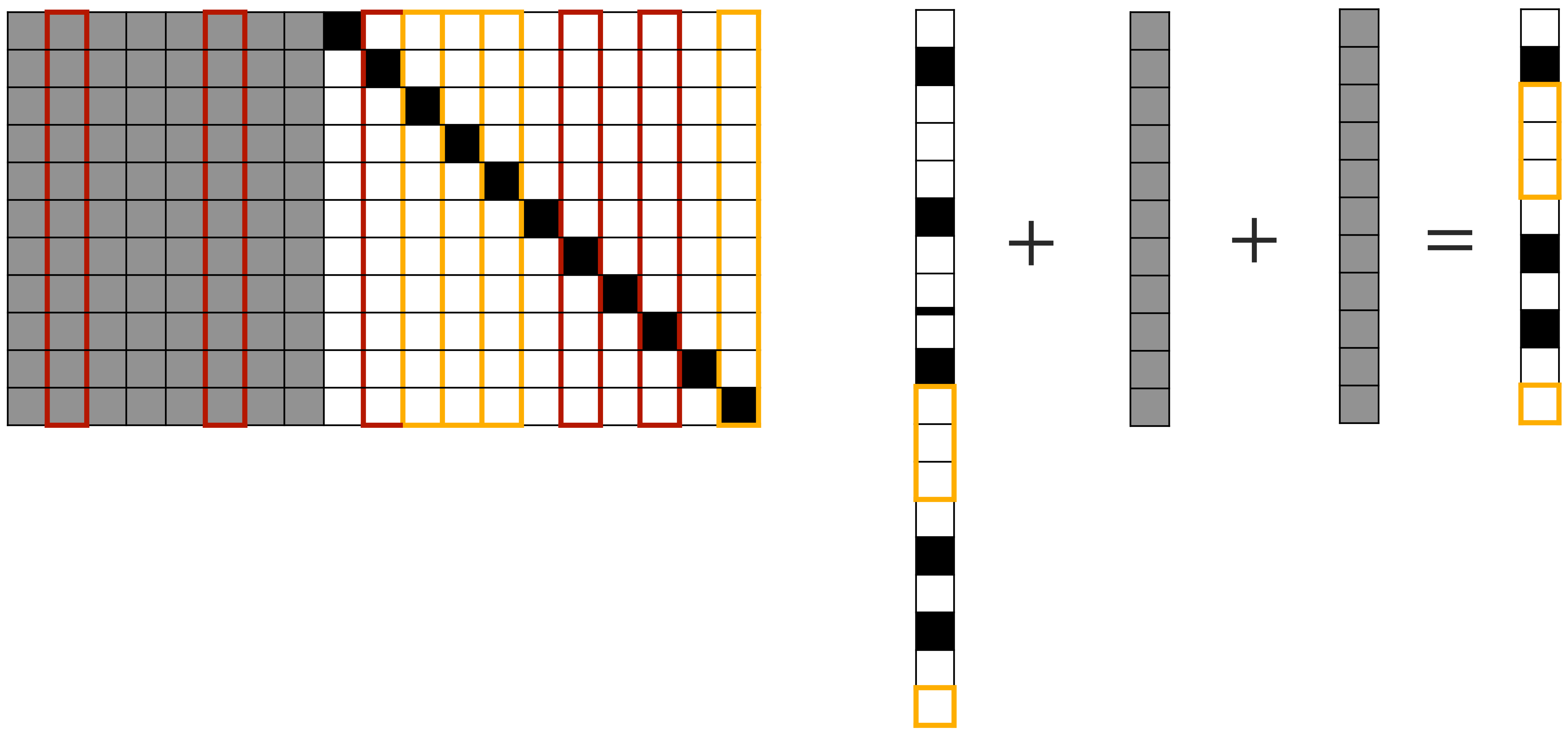
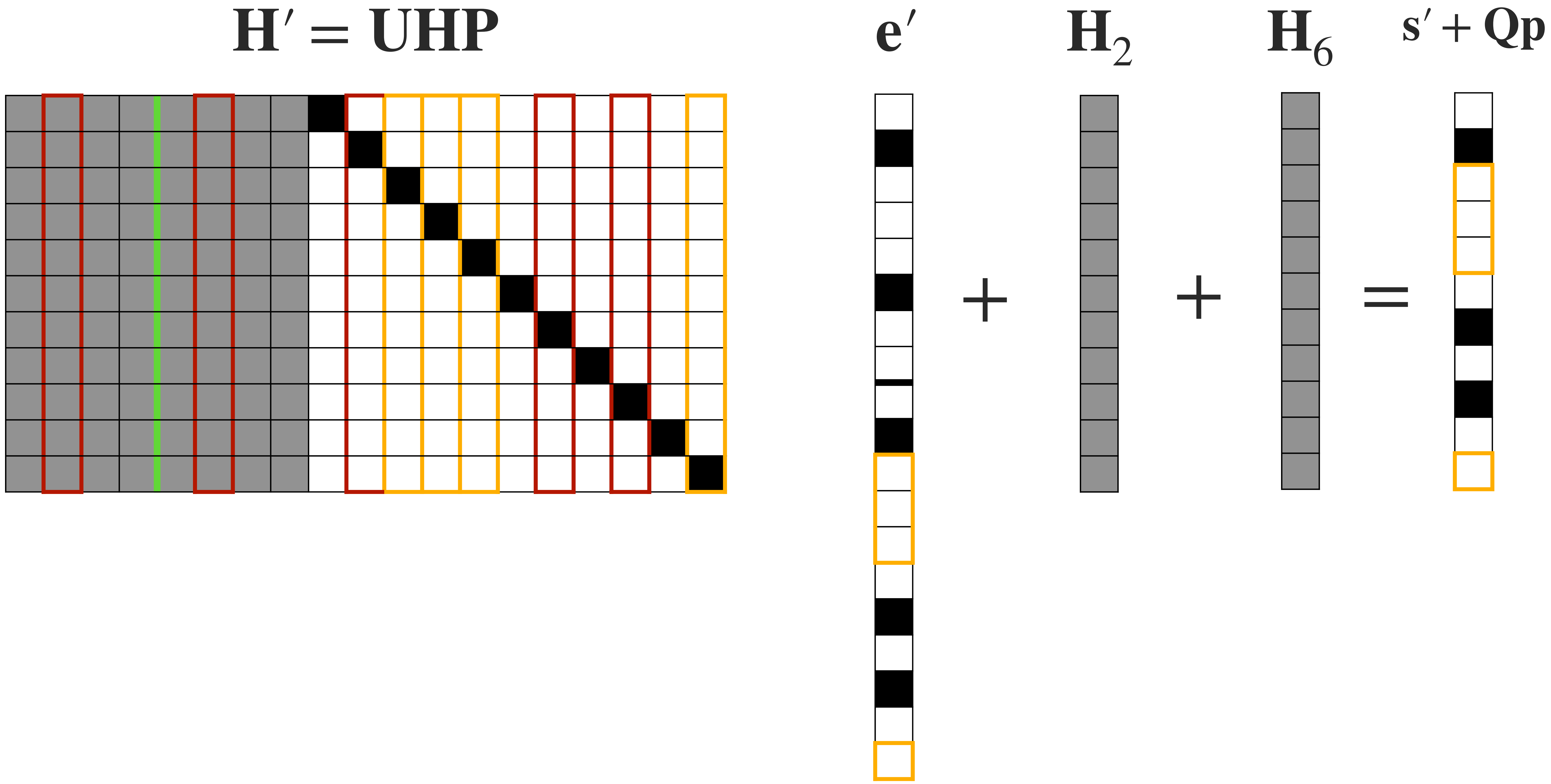
Leon's attack: complexity

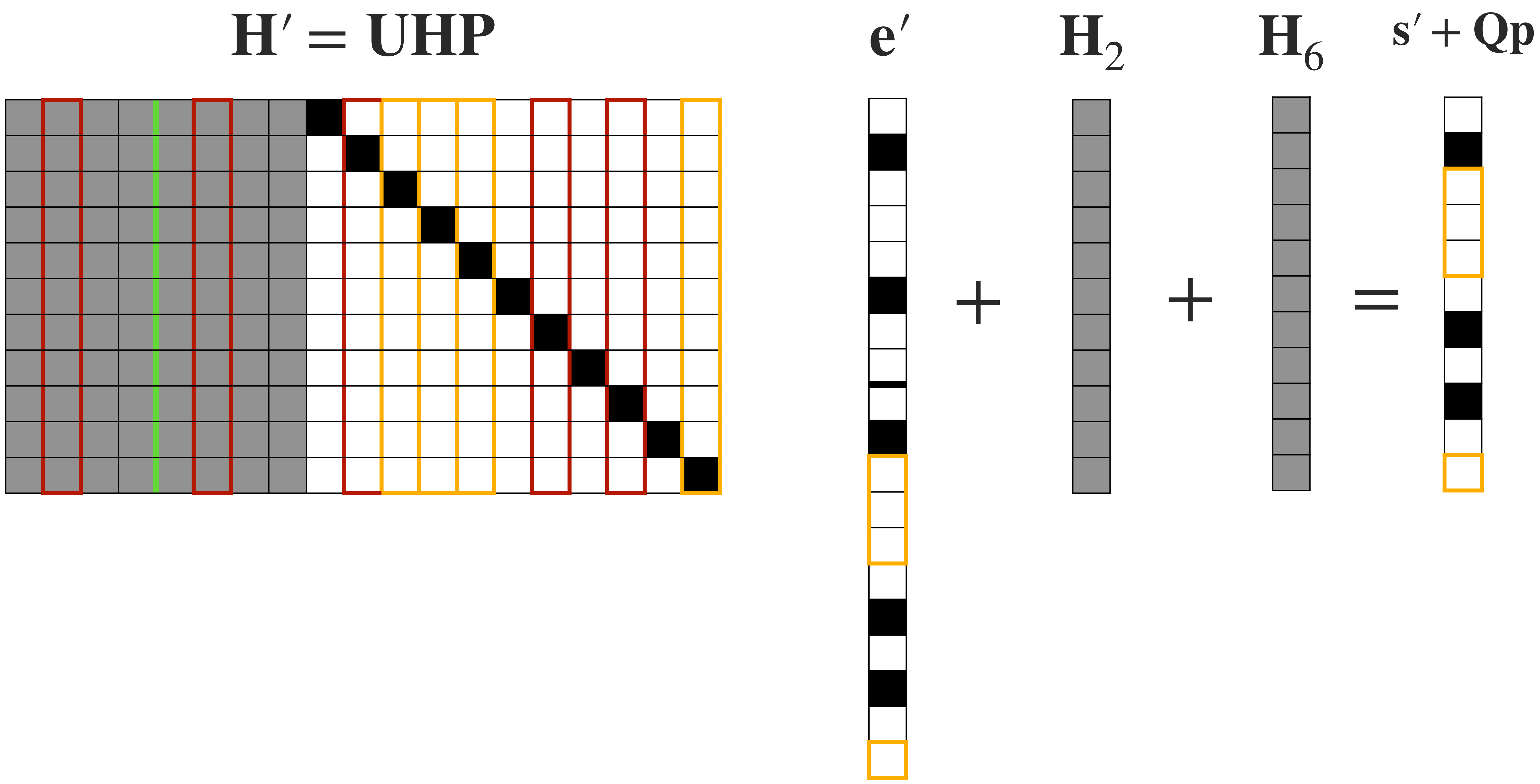


Probability that we are in the correct configuration: $$\frac{\binom{n-k-l}{t-p}\binom{k}{p}}{\binom{n}{t}}.$$

Stern's attack

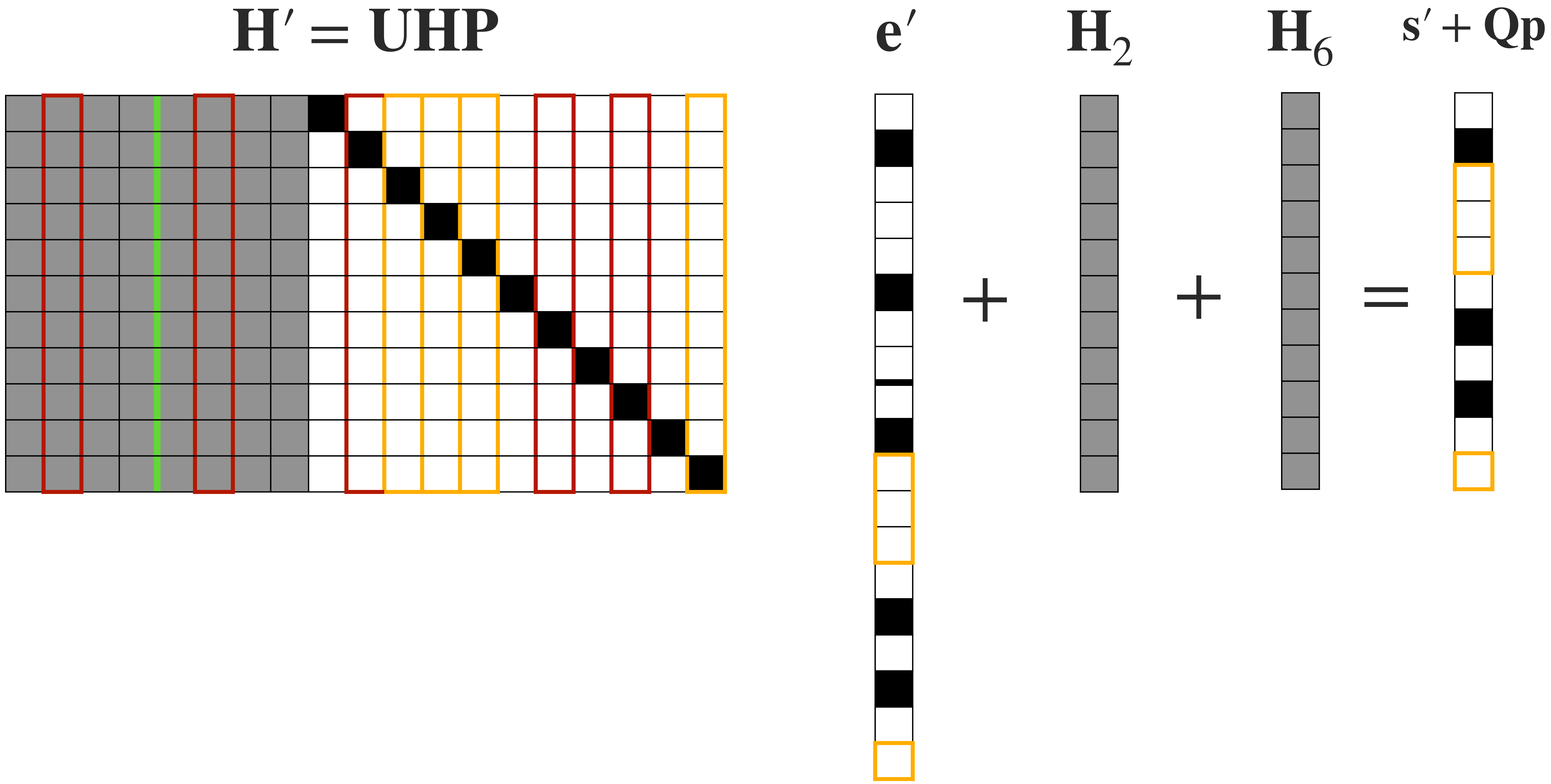


Stern's attack



Suppose that there are exactly $\frac{p}{2}$ errors in the first half of $\mathbf{Q}$ and exactly $\frac{p}{2}$ errors in the first half of $\mathbf{Q}$.

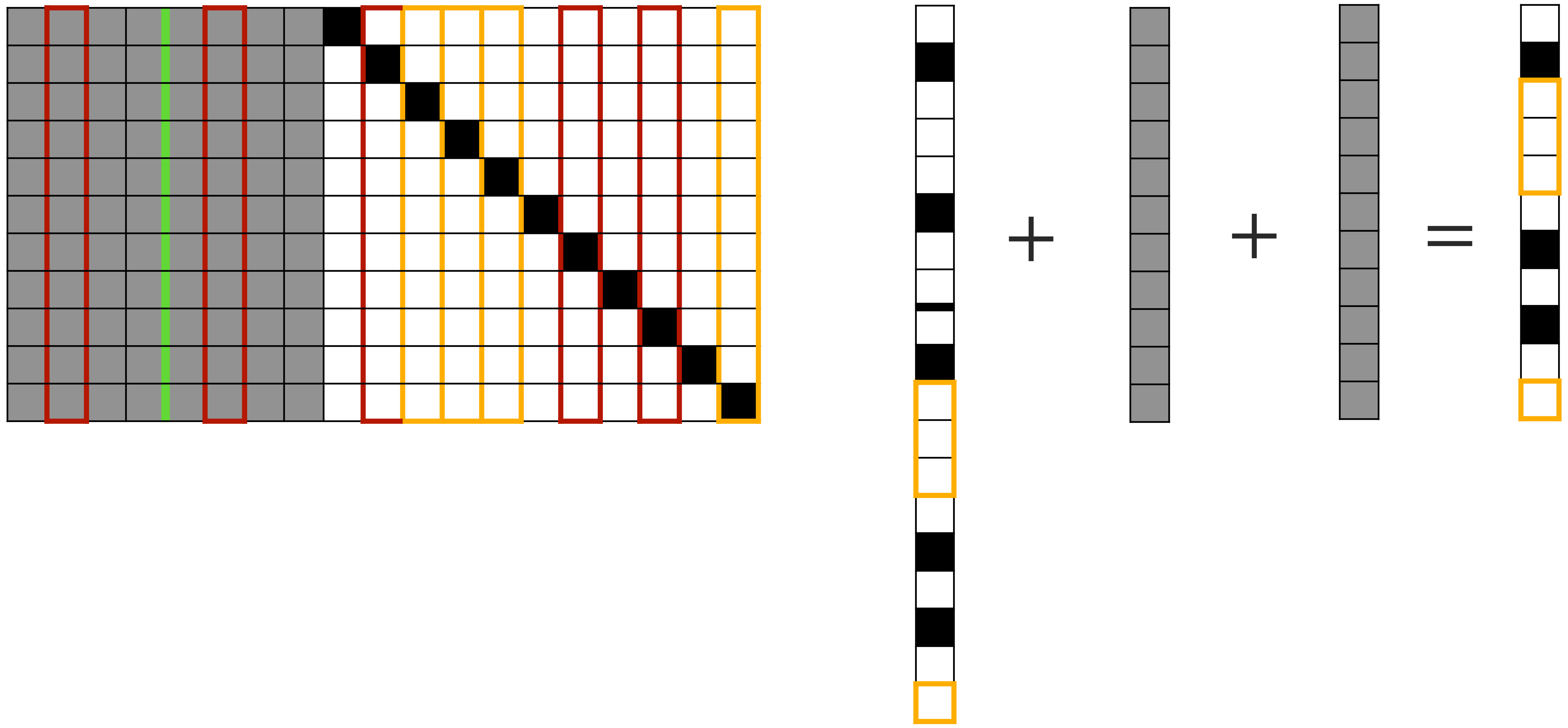
Stern's attack



Suppose that there are exactly $\frac{p}{2}$ errors in the first half of $\mathbf{Q}$ and exactly $\frac{p}{2}$ errors in the first half of $\mathbf{Q}$.

Instead of looking for an all zero subset of rows, we are looking for a collision.

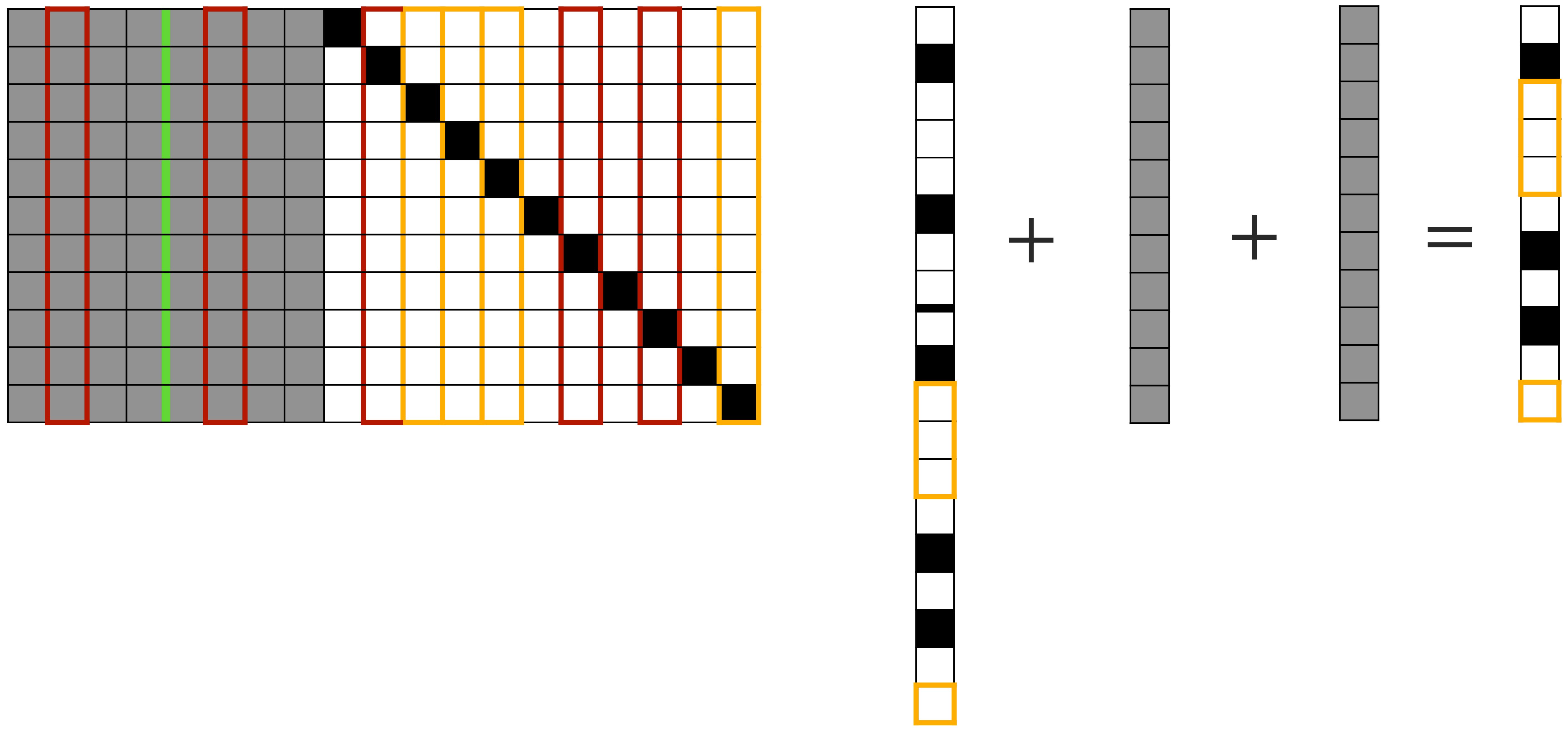
Stern's attack



- Pick a subset L of l rows: $\mathbf{H}_L$.
- Permute $\mathbf{H}$ and bring to systematic form (then $\mathbf{H}_L = \begin{pmatrix}\mathbf{Q}_L & \mathbf{I}_L\end{pmatrix}$).
- Split $\mathbf{Q}$ into two disjoint parts: $\mathbf{Q} = (\mathbf{A} \quad \mathbf{B})$.
- Build a list of vectors $(\mathbf{s}'_L + \mathbf{A}_L\mathbf{a})$ for all (many) $\mathbf{a}$.
- For all (many) $\mathbf{b}$:
  - If $\mathbf{B}_L\mathbf{b}$ collides with (is equal to) any of the vectors in the list built in the fourth step
    - If $\mathrm{wt}(\mathbf{s}' + \mathbf{Aa} + \mathbf{Bb}) = t - p$ then put $\mathbf{e}' = \mathbf{a}||\mathbf{b}||(\mathbf{s}' + \mathbf{Aa} + \mathbf{Bb})$. Output unpermuted version of $\mathbf{e}$.
- Else return to the second step and rerandomize.

a and **b** are vectors chosen from $W = \{\mathbf{w} \in \mathbb{F}_2^{k/2} \,|\, \mathrm{wt}(\mathbf{w}) = \frac{p}{2}\}$

Stern's attack: complexity



Probability that we are in the correct configuration: $$\frac{\binom{n-k-l}{t-p}\binom{k}{\frac{p}{2}}^2}{\binom{n}{t}}.$$

ISD algorithms summary

Prange

k	$n-k$
0	t

Lee-Brickell

k	$n-k$
p	$t-p$

Leon

k	l	$n-k-l$
p	0	$t-p$

Stern

$k/2$	$k/2$	l	$n-k-l$
$p/2$	$p/2$	0	$t-p$

Next time:

MPC-in-the-Head construction