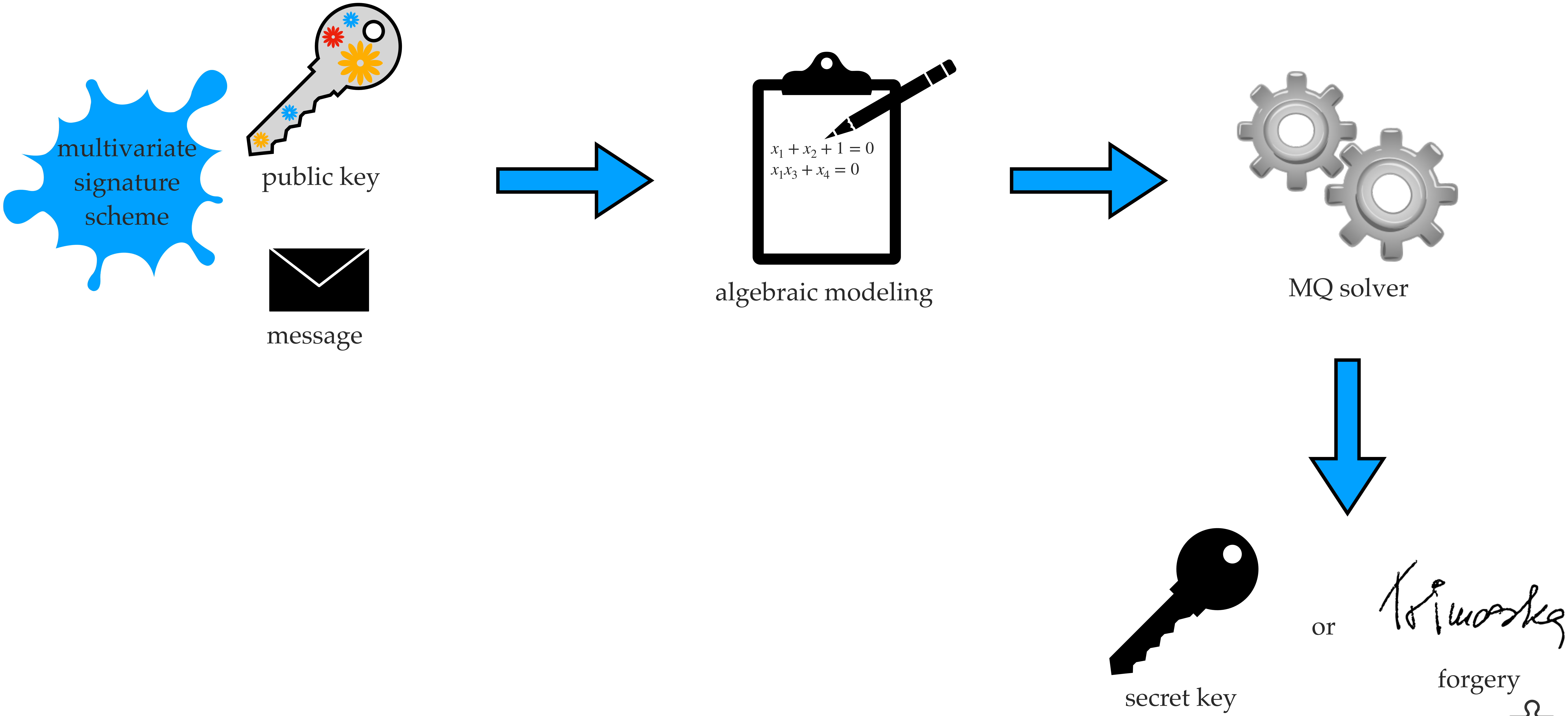
Multivariate cryptography

Monika Trimoska

Selected Areas in Cryptology - Part 1
Spring, 2024

TU/e

Algebraic cryptanalysis (recall)



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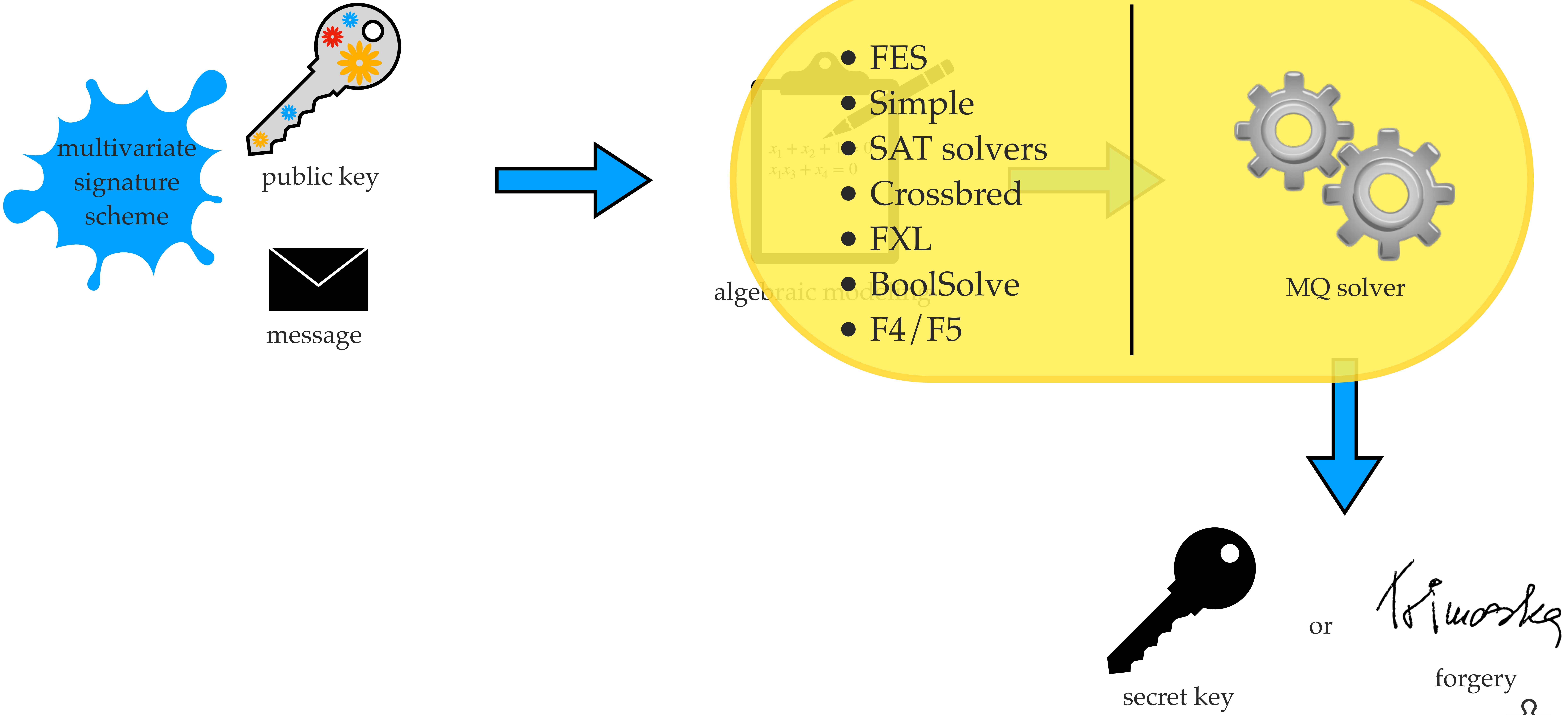

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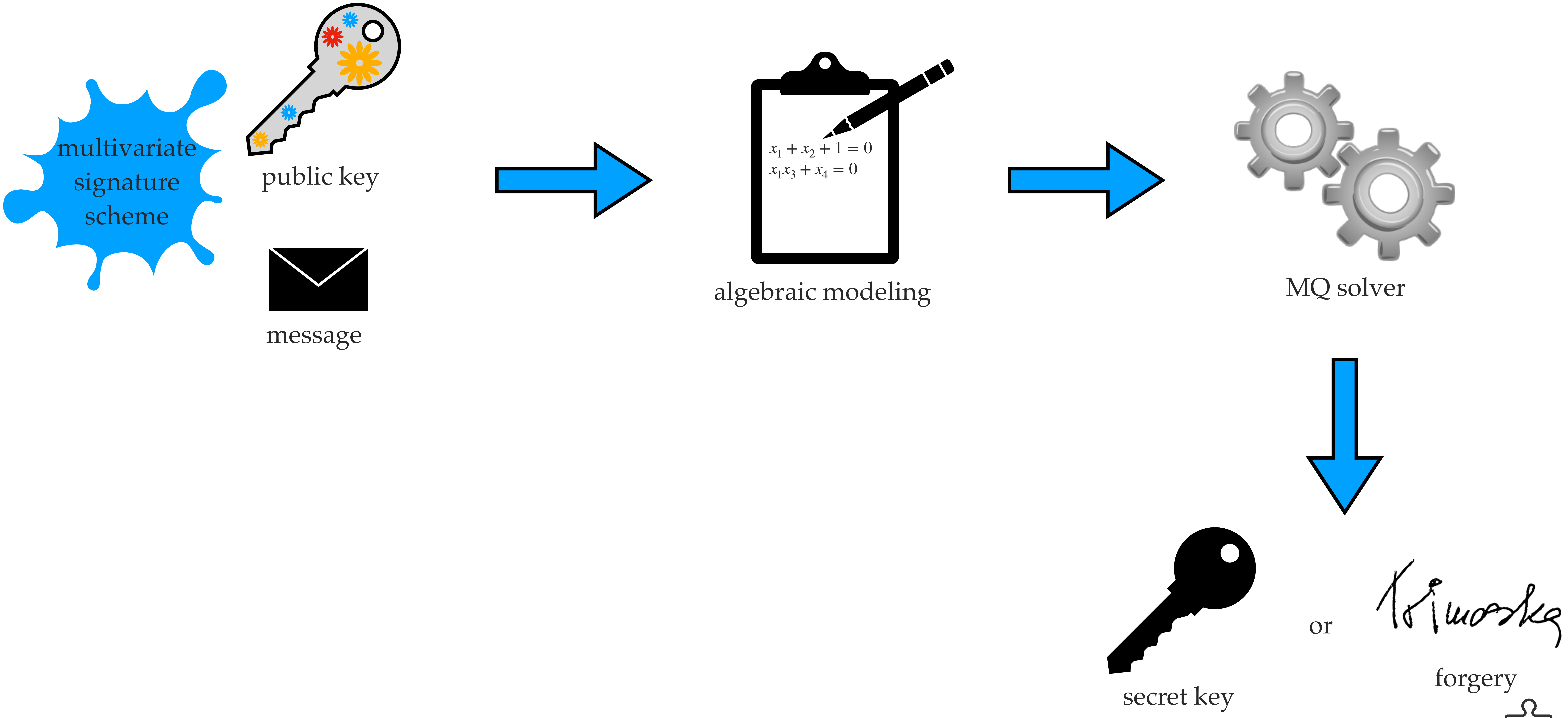

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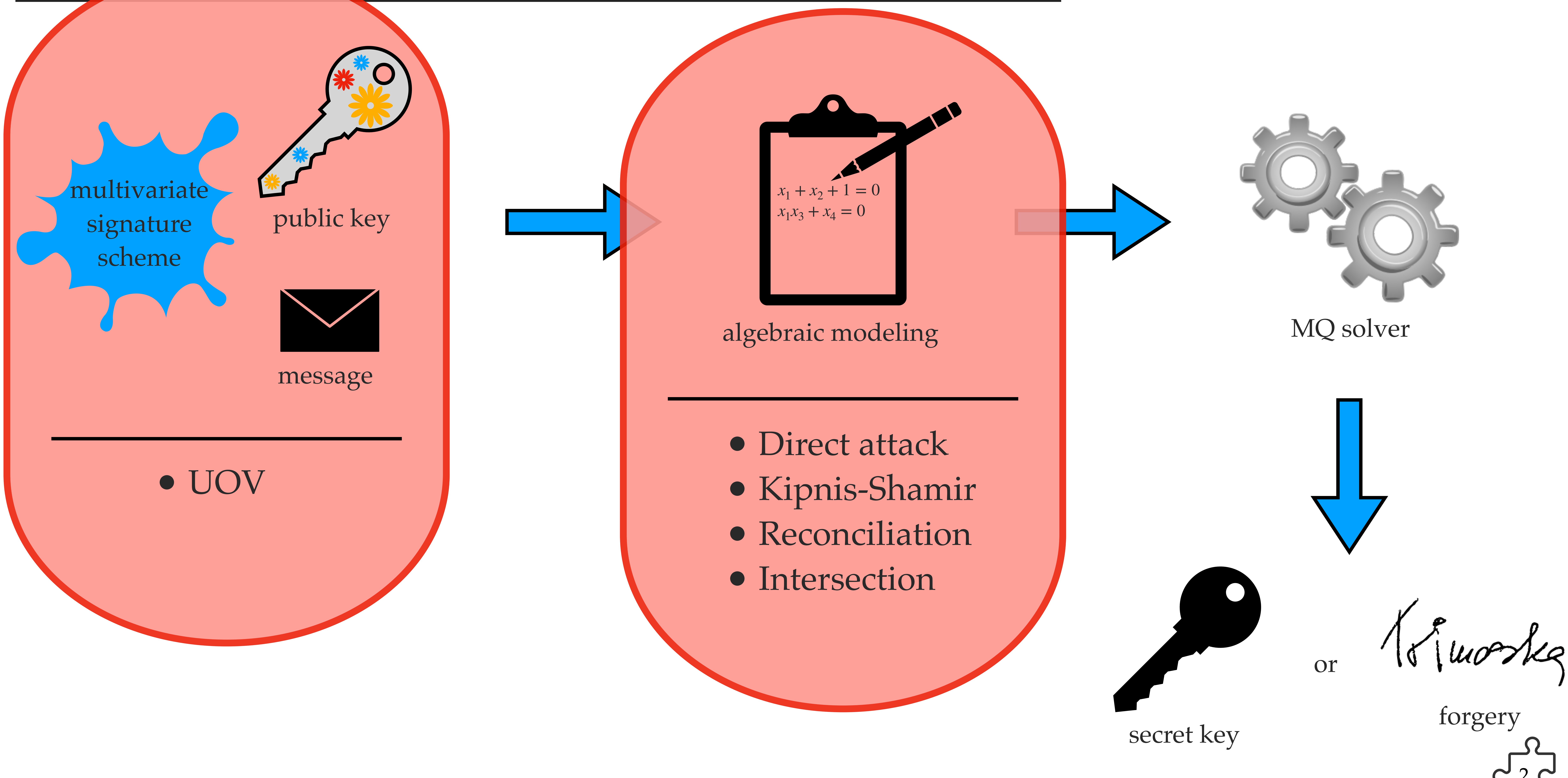

Modelisation

A motivating example.

Given matrices $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2 \in \mathscr{M}_{n,n}(\mathbb{F}_q)$ (the space of matrices over $\mathbb{F}_q$ of size $n \times n$), find $\mathbf{A}, \mathbf{B} \in \mathrm{GL}_n(\mathbb{F}_q)$ (the space of invertible matrices over $\mathbb{F}_q$ of size $n \times n$), such that

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→ Demo

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→ Demo

→ In the assignment:
- Write down the equations;
- Find a better modelisation for this problem;

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→ Results in a linear system with the same number of variables and equations.

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- Demo
- Results in a linear system with the same number of variables and equations.
  - If $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2$ are all full rank, we should have a unique solution.
  - We can easily recover $\mathbf{A}$ from $\mathbf{A}^{-1}$.

Multivariate digital signature schemes

Multivariate signatures

Fiat-Shamir construction

trapdoor construction

Examples.

MQDSS

SOFIA

Examples.

HFEv-

UOV

The MQ problem (recall)

A quadratic system of m equations in n variables over a finite field $\mathbb{F}_q$:

$$f^{(k)}(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} \gamma_{ij}^{(k)} x_i x_j + \sum_{1 \le i \le n} \beta_i^{(k)} x_i + \alpha^{(k)}$$

The MQ problem

Given m multivariate quadratic polynomials $f^{(1)}, \ldots, f^{(m)}$ of n variables over a finite field $\mathbb{F}_q$, find a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$, such that $f^{(1)}(\mathbf{x}) = \ldots = f^{(m)}(\mathbf{x}) = 0$.

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→ Hard in general (should be hard for randomly generated instances).

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→ Hard in general (should be hard for randomly generated instances).

→ Can become easy if we have some structure (a trapdoor).

The trapdoor construction

The trapdoor construction

- Central map:
  $f : (x_1, \ldots, x_n) \in \mathbb{F}_q^n \to \left(f^{(1)}(x_1, \ldots, x_n), \ldots, f^{(m)}(x_1, \ldots, x_n)\right) \in \mathbb{F}_q^m$

- Two bijective linear (or affine) transformations:
  $\mathbf{S} \in \mathrm{GL}_n(\mathbb{F}_q)$ and $\mathbf{T} \in \mathrm{GL}_m(\mathbb{F}_q)$

- Public map:
  $p = \mathbf{T} \circ f \circ \mathbf{S}$

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Main idea:

- The central map has a structure such that it is easy to find preimages: it is easy (polynomial time) to compute $f^{-1}(\mathbf{x})$ for a target vector $\mathbf{x}$.

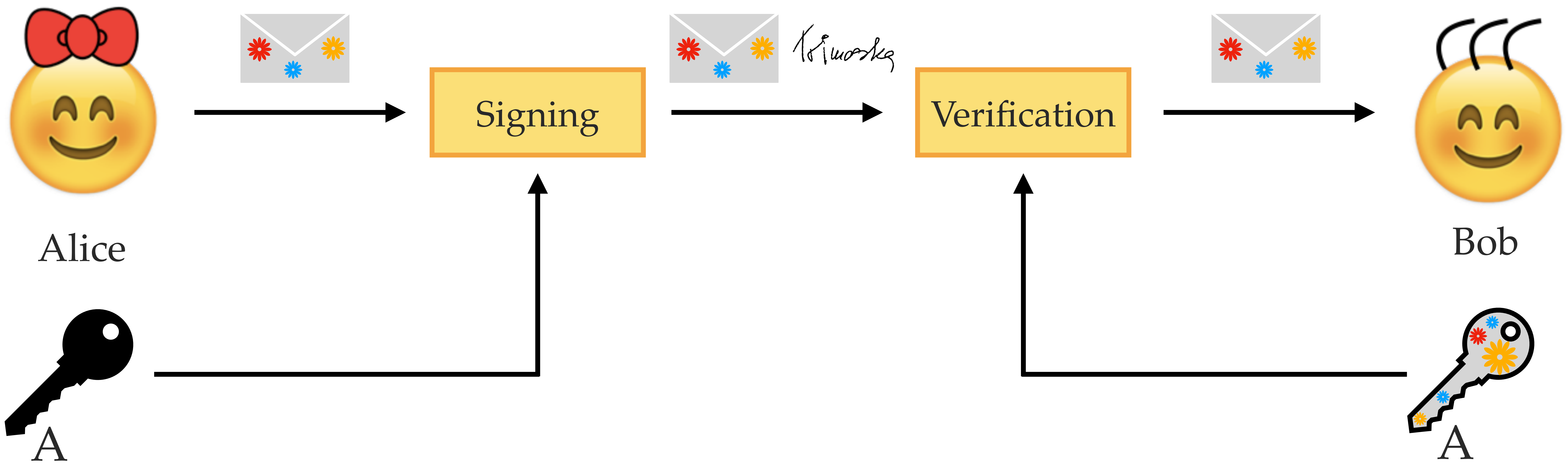
- The linear transformations hide the structure of the central map.

The trapdoor construction

$$\begin{tikzcd}
\mathbf{w} \in \mathbb{F}_q^m \arrow[r, "\mathbf{T}^{-1}"] & \mathbf{x} \in \mathbb{F}_q^m \arrow[r, "f^{-1}"] & \mathbf{y} \in \mathbb{F}_q^n \arrow[r, "\mathbf{S}^{-1}"] & \mathbf{z} \in \mathbb{F}_q^n \arrow[lll, bend left, "p"]
\end{tikzcd}$$

General workflow

The trapdoor construction



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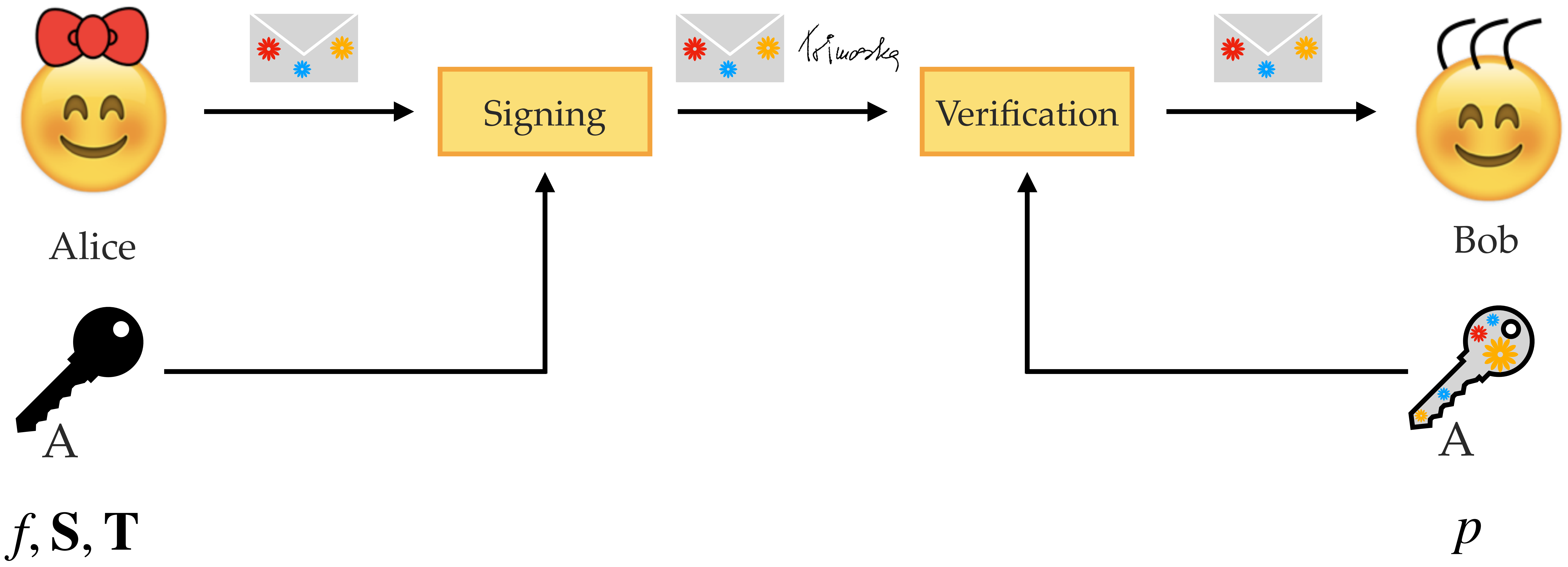

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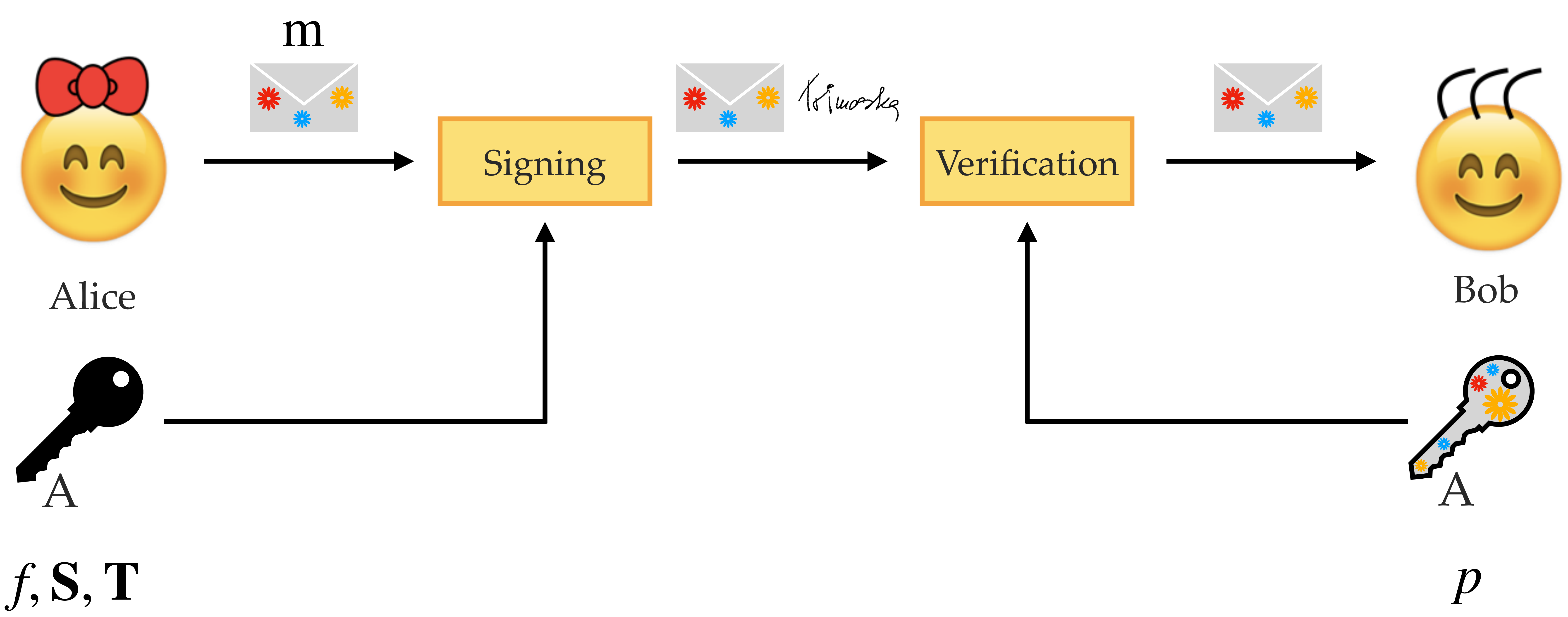

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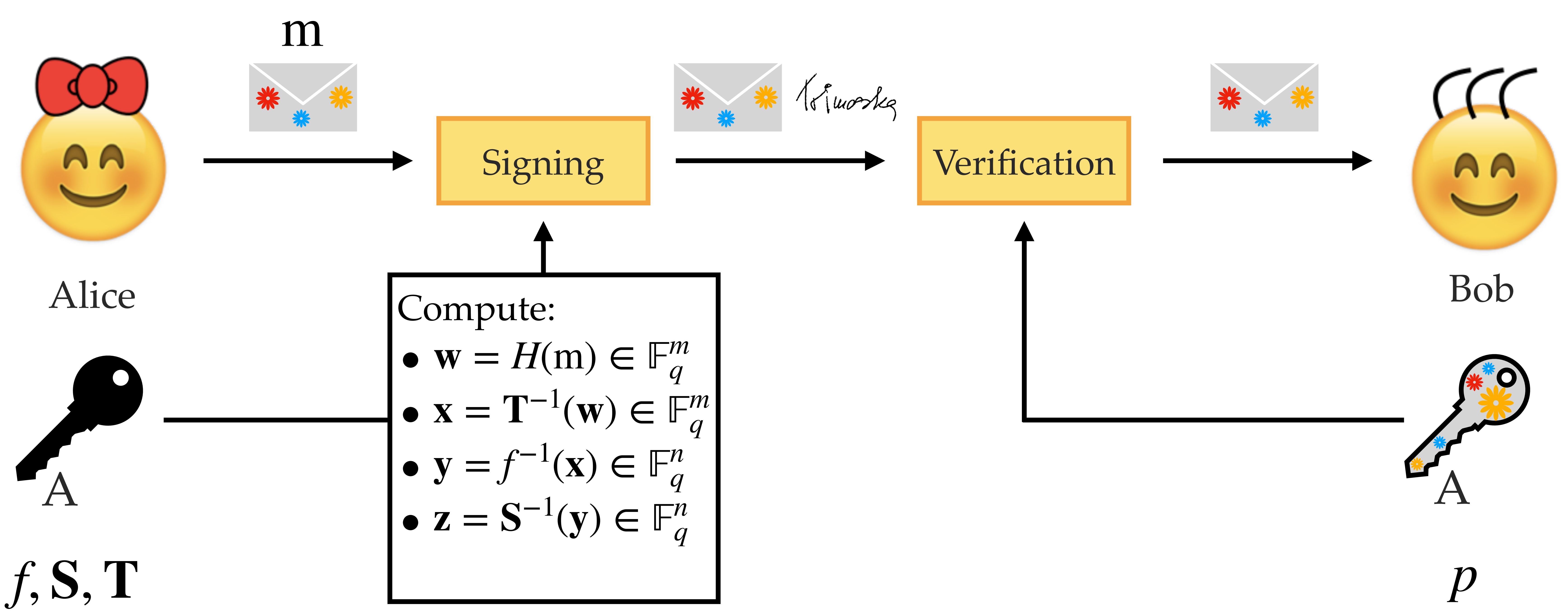

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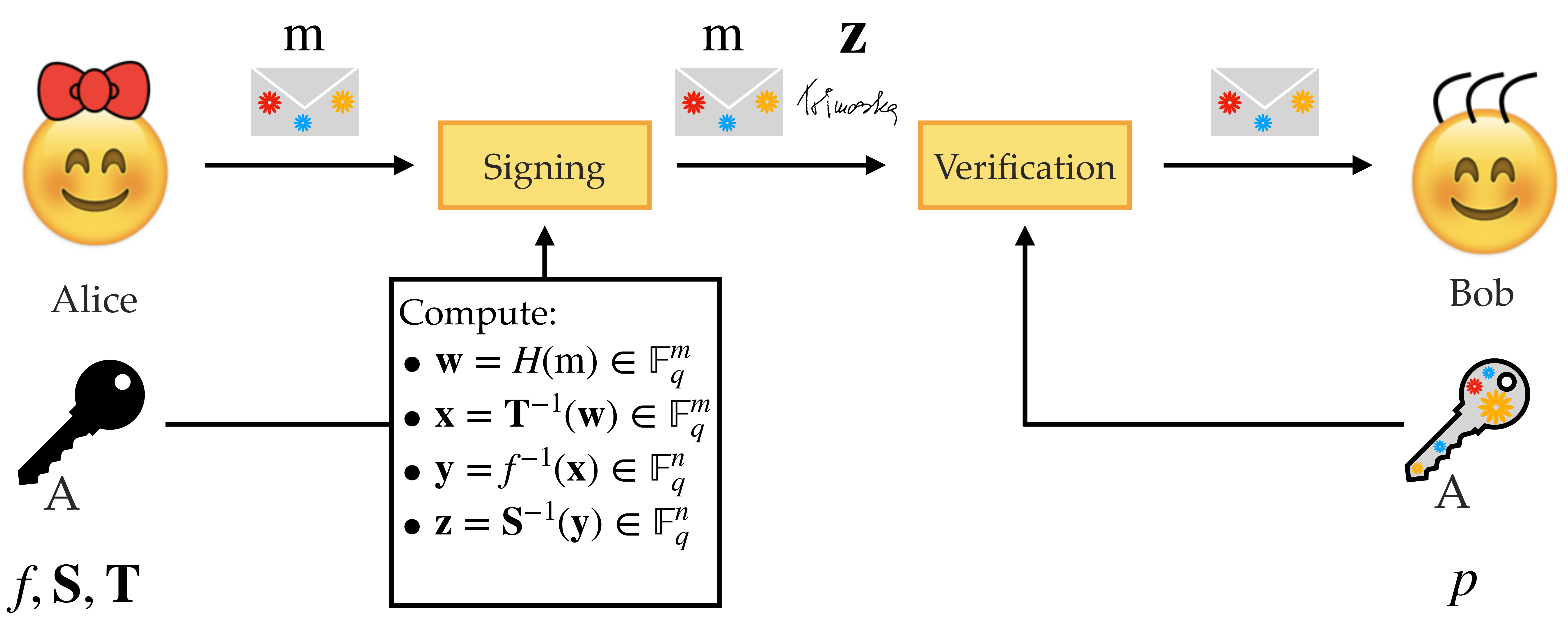

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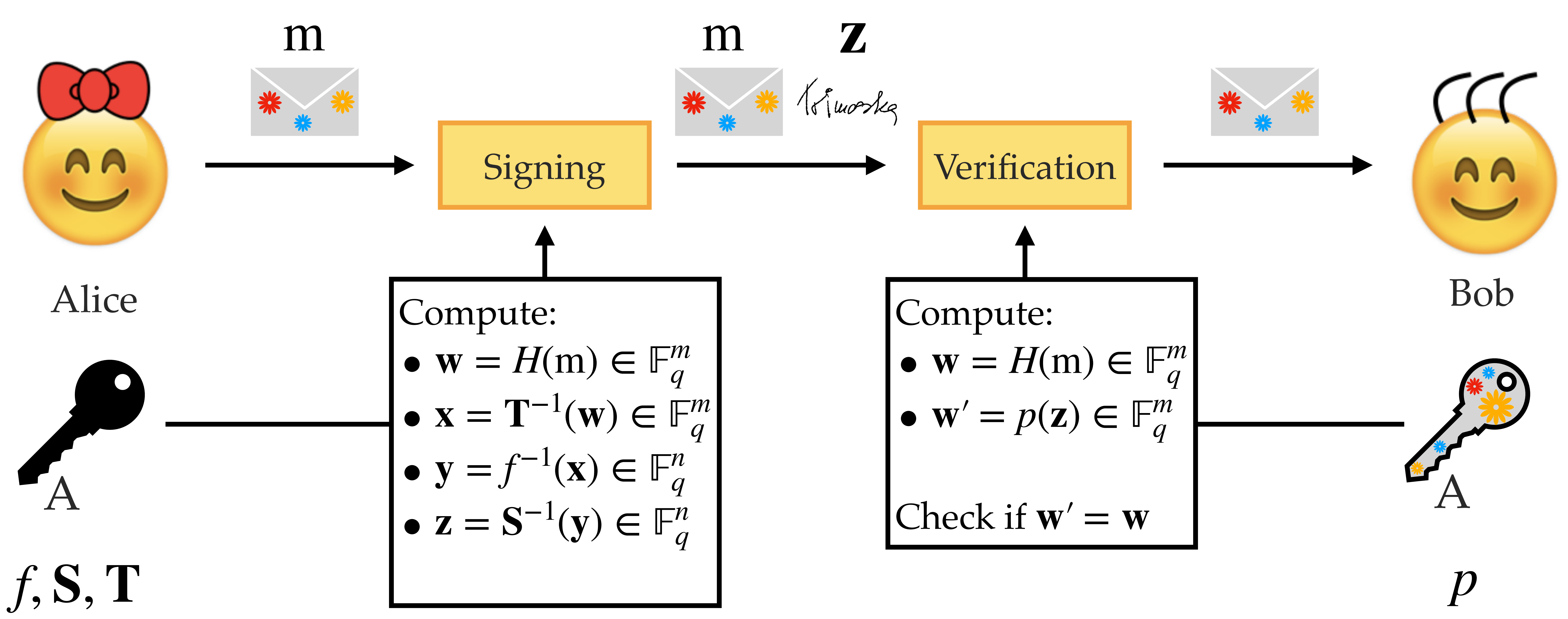

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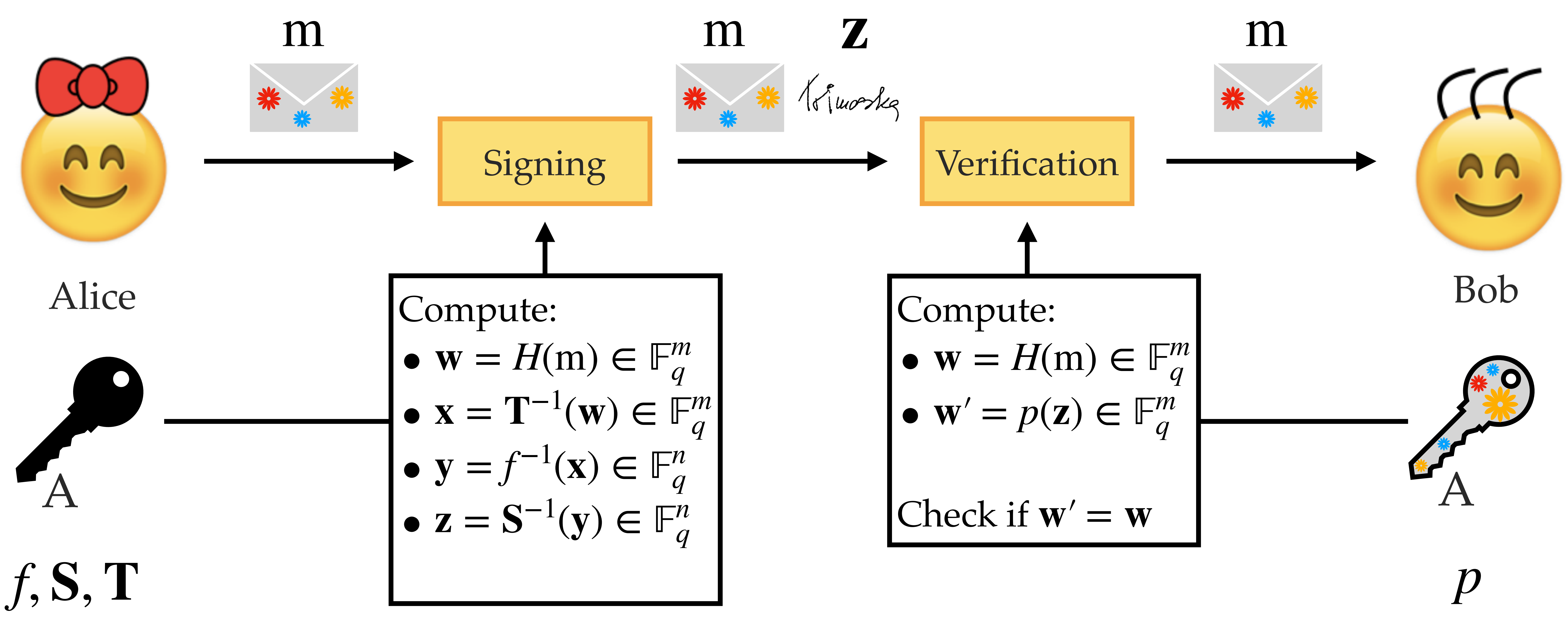

Isomorphism of polynomials

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The Isomorphism of Polynomials (IP) problem

Input: Two m-tuples of multivariate polynomials
$f = (f^{(1)}, \ldots, f^{(m)}), p = (p^{(1)}, \ldots, p^{(m)}) \in \mathbb{F}_q[x_1, \ldots, x_n]^m$.
Question: Find - if any - $\mathbf{S} \in \mathrm{GL}_n(\mathbb{F}_q)$ and $\mathbf{T} \in \mathrm{GL}_m(\mathbb{F}_q)$ such that $p = \mathbf{T} \circ f \circ \mathbf{S}$.

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The Extended Isomorphism of Polynomials (EIP) problem

Input: An m-tuple of multivariate polynomials $p = (p^{(1)}, \dots, p^{(m)}) \in \mathbb{F}_q[x_1, \dots, x_n]^m$ and a special class of m-tuples of multivariate polynomials $\mathscr{C} \subseteq \mathbb{F}_q[x_1, \dots, x_n]^m$.
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Signature schemes with the trapdoor construction rely on EIP, because we do not have the central map f, but we know the special class to which it belongs (example - UOV - coming up).

Unbalanced Oil and Vinegar
(UOV)

The UOV central map

Unbalanced Oil and Vinegar [Kipnis, Patarin, Goubin, '99]

$$f^{(k)}(x_1, \ldots, x_n) = \sum_{i \in V, j \in V} \gamma_{ij}^{(k)} x_i x_j + \sum_{i \in V, j \in O} \gamma_{ij}^{(k)} x_i x_j + \sum_{i=1}^{n} \beta_i^{(k)} x_i + \alpha^{(k)}$$

Index set of vinegar variables: $V = \{1, \ldots, v\}$

Index set of oil variables: $O = \{v+1, \ldots, n\}$

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- The central map is constructed in such a way that enumerating all of the vinegar variables leaves us with a linear system in the oil variables (oil does not mix with oil).
- Everything is as described in the previous slides, except that we do not have a linear transformation on the output: $\mathbf{T} = \mathbf{I}$.

Matrix representation of quadratic forms

Quadratic form: $f(\mathbf{x}) = \sum \gamma_{ij} x_i x_j$

$$\mathbf{x}^\top = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \gamma_{1,1} & \frac{\gamma_{1,2}}{2} & \frac{\gamma_{1,3}}{2} & \frac{\gamma_{1,4}}{2} \\ \frac{\gamma_{2,1}}{2} & \gamma_{2,2} & \frac{\gamma_{2,3}}{2} & \frac{\gamma_{2,4}}{2} \\ \frac{\gamma_{3,1}}{2} & \frac{\gamma_{3,2}}{2} & \gamma_{3,3} & \frac{\gamma_{3,4}}{2} \\ \frac{\gamma_{4,1}}{2} & \frac{\gamma_{4,2}}{2} & \frac{\gamma_{4,3}}{2} & \gamma_{4,4} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

so with $\mathbf{x} = (x_1, \ldots, x_n)$, we get $\mathbf{x}^\top \mathbf{F} \mathbf{x}$.

Matrix representation of bilinear forms

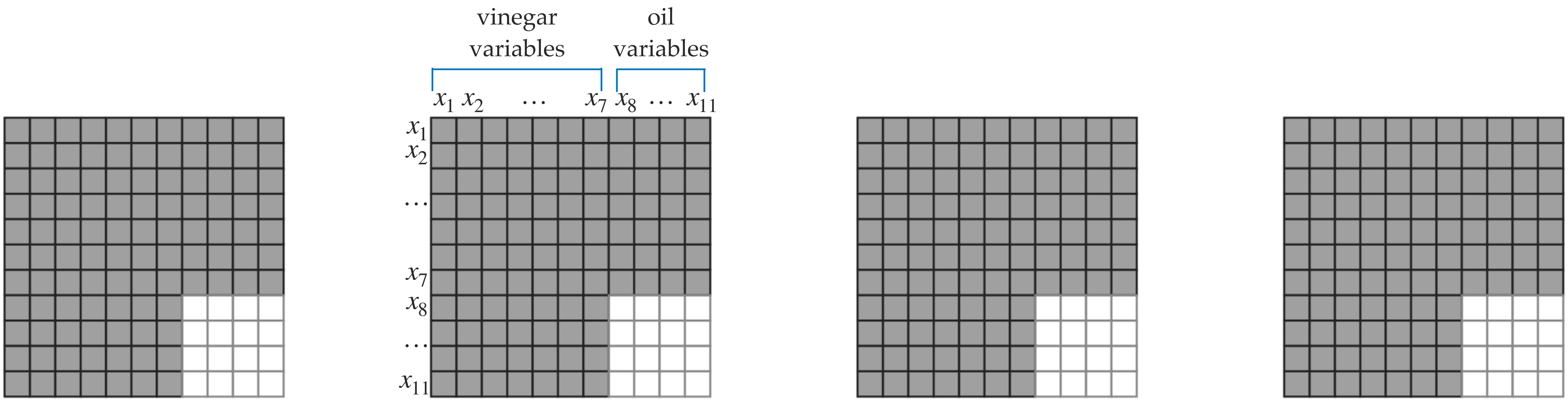
Bilinear form: $f(\mathbf{x}, \mathbf{y}) = \sum \gamma_{ij} x_i y_j$

$$\underbrace{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}}_{\mathbf{x}^\top} \quad \underbrace{\begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} & \gamma_{1,4} \\ \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \gamma_{2,4} \\ \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \gamma_{3,4} \\ \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,3} & \gamma_{4,4} \end{bmatrix}}_{\mathbf{B}} \quad \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{\mathbf{y}}$$

so with $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, we get $\mathbf{x}^\top \mathbf{B} \mathbf{y}$.

The UOV central map

Toy example: $v = 7, m = 4$



$\mathbf{F}^{(1)}$ $\mathbf{F}^{(2)}$ $\mathbf{F}^{(3)}$ $\mathbf{F}^{(4)}$

*Grayed areas represent the entries that are possibly nonzero; blank areas denote the zero entries;

UOV key generation

In matrix representation

$$\mathbf{P}^{(k)} = \mathbf{S}^{\top}\mathbf{F}^{(k)}\mathbf{S}, \text{ for all } k \in \{1, \ldots, m\}.$$

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Why ?

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In matrix representation

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By definition, $p = f \circ \mathbf{S}$.

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$$\mathbf{x}^\top \mathbf{P}^{(k)} \mathbf{x} = (\mathbf{S}\mathbf{x})^\top \mathbf{F}^{(k)} (\mathbf{S}\mathbf{x})$$

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UOV in the NIST competition

UOV
TUOV
PROV
MAYO
VOX
QR-UOV
SNOVA

UOV in the NIST competition

- UOV
- TUOV
- PROV
- MAYO
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- SNOVA

Example.

	NIST SL	n	m	$\mathbb{F}_q$	$\lvert\mathsf{pk}\rvert$ (bytes)	$\lvert\mathsf{sk}\rvert$ (bytes)	$\lvert\mathsf{cpk}\rvert$ (bytes)	$\lvert\mathsf{sig+salt}\rvert$ (bytes)
`ov-Ip`	1	112	44	$\mathbb{F}_{256}$	278 432	237 896	43 576	128
`ov-Is`	1	160	64	$\mathbb{F}_{16}$	412 160	348 704	66 576	96
`ov-III`	3	184	72	$\mathbb{F}_{256}$	1 225 440	1 044 320	189 232	200
`ov-V`	5	244	96	$\mathbb{F}_{256}$	2 869 440	2 436 704	446 992	260

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- We choose $n \sim 2.5m$ (slightly bigger than)

UOV-like schemes have:

- Big public keys
- Small signatures

Attacks on UOV

O v

Attacks on UOV

- Direct attack
- Reconciliation attack
- Kipnis-Shamir attack
- Intersection attack

Direct attack

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Try to forge a signature with only the knowledge of the public key.

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Constraint for modelisation

For a target $\mathbf{w}$, find $\mathbf{z}$ such that $p(\mathbf{z}) = \mathbf{w}$.

Direct attack

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Constraint for modelisation

For a target $\mathbf{w}$, find $\mathbf{z}$ such that $p(\mathbf{z}) = \mathbf{w}$.

Equations:

$$\begin{aligned}
\mathbf{z}^\top \mathbf{P}^{(1)} \mathbf{z} &= w_1 \\
\mathbf{z}^\top \mathbf{P}^{(2)} \mathbf{z} &= w_2 \\
&\dots \\
\mathbf{z}^\top \mathbf{P}^{(m)} \mathbf{z} &= w_m
\end{aligned}$$

Reconciliation attack

The secret subspace O

The map p with a UOV trapdoor vanishes on a linear subspace $O \subset \mathbb{F}_q^n$ of $\dim(O) = m$:

$$p(\mathbf{o}) = 0, \text{ for all } \mathbf{o} \in O.$$

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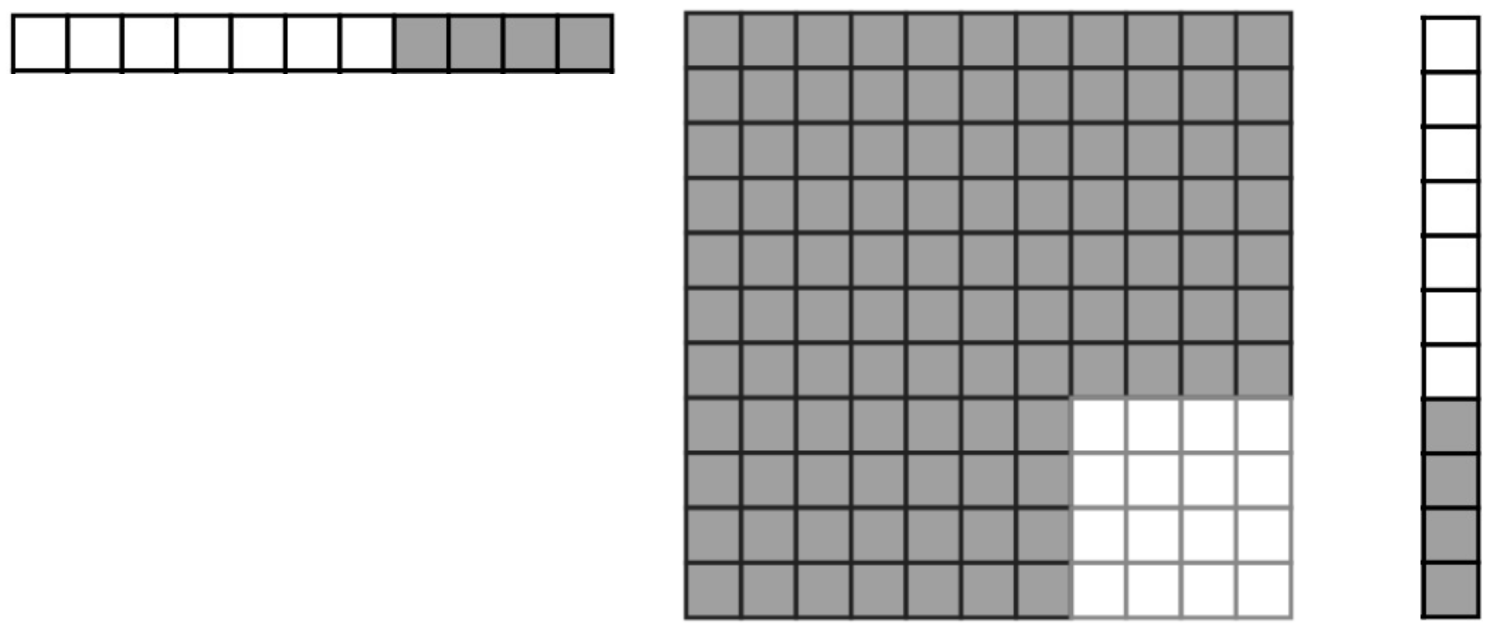
Why ?

Let $O' \in \mathbb{F}_q^n$ be the m-dimensional space that consists of all the vectors whose first $n - m$ entries (corresponding to the vinegar variables) are zero: $O' = \{\mathbf{v} \,|\, v_i = 0 \text{ for all } i \leq n - m\}$.



$= 0$

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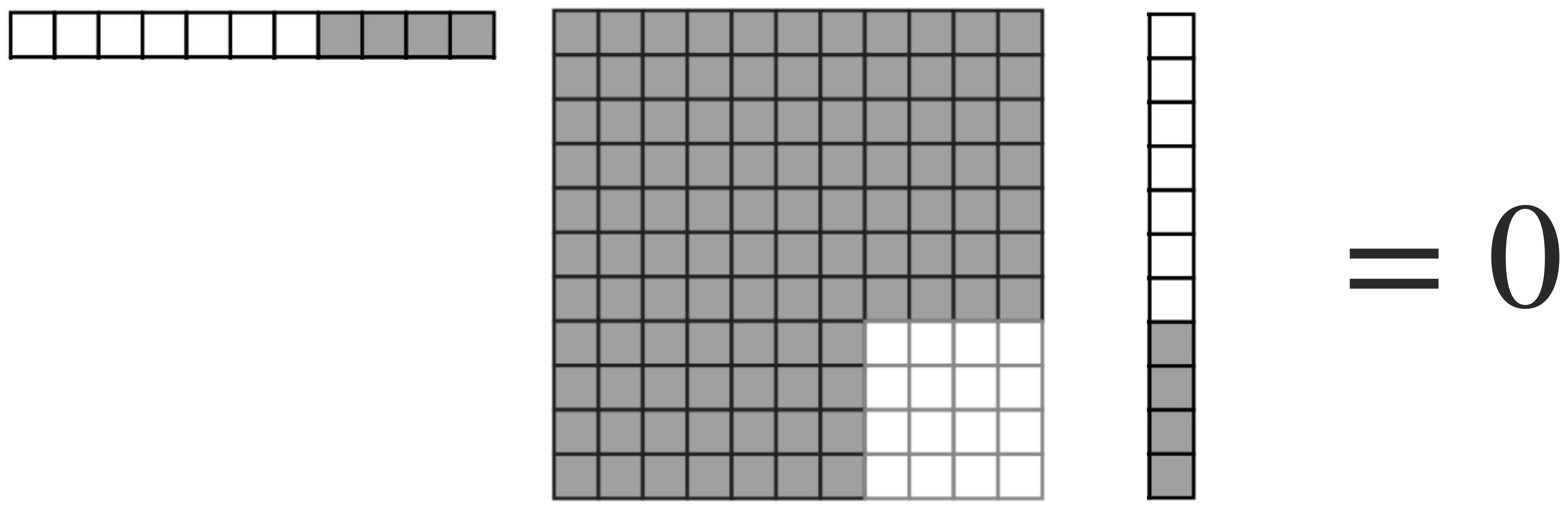

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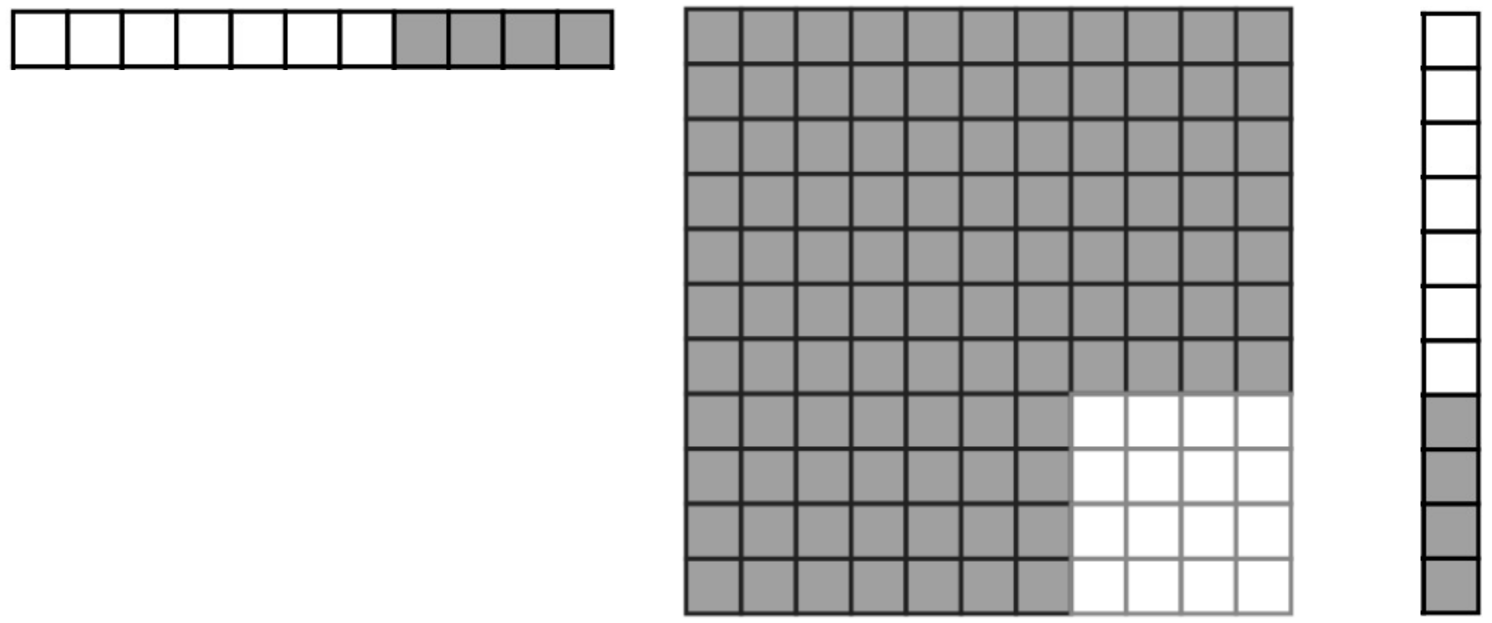

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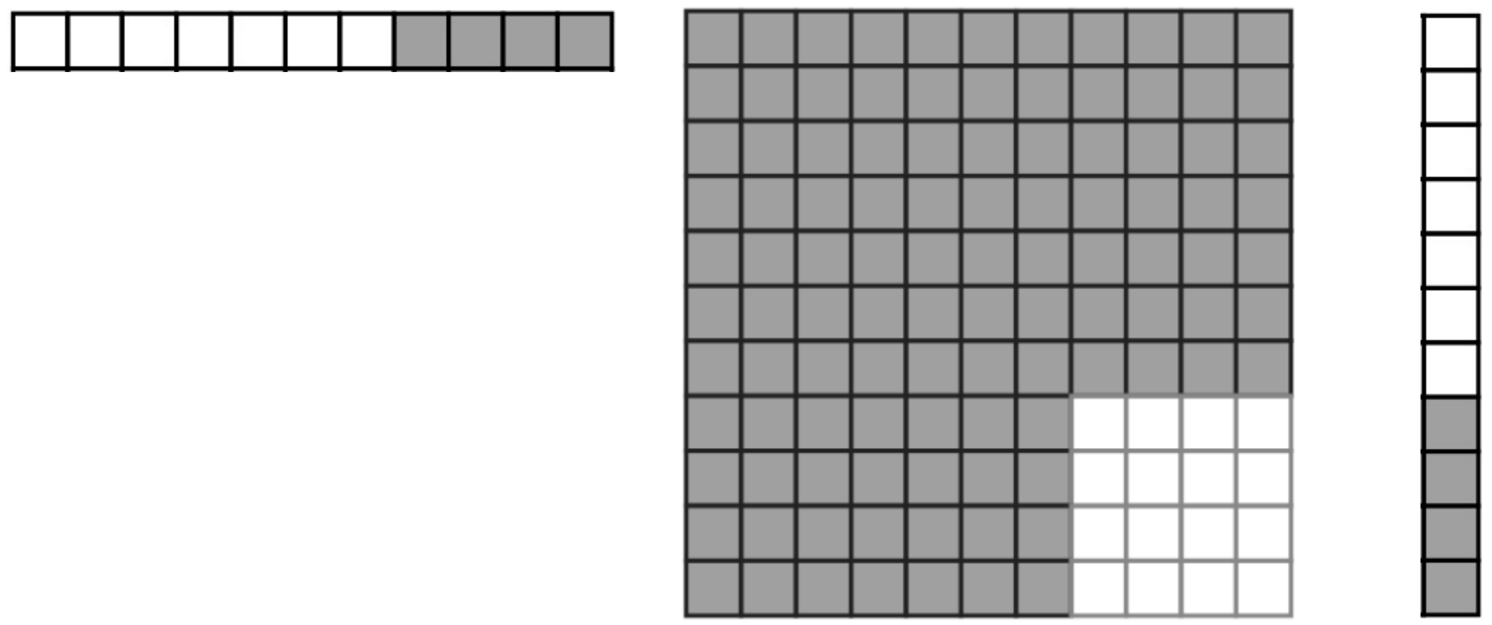

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Reconciliation attack

Find the secret oil subspace O : find m linearly independent vectors in O.

The polar form

The polar form of a quadratic map $p = (p^{(1)}, \ldots, p^{(m)})$ is the bilinear form $p' = (p'^{(1)}, \ldots, p'^{(m)})$ such that

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$\longrightarrow$ So, p' is bilinear and symmetric.

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Equations:

For $i \in \{1,\ldots,m\}$ **do**

$\mathbf{o}_i = (o_1, \ldots, o_v, 0, \ldots, 1_{n-i+1}, 0, \ldots, 0)$

Solve:

$$\mathbf{o}_i^\top \mathbf{B}^{(k)} \mathbf{o}_j = 0, \text{ for } k \in \{1,\ldots,m\} \text{ and } j < i$$

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In the first iteration, we have only quadratic equations, so this is the bottleneck. Linear constraints facilitate the resolution of a system.

Kipnis-Shamir attack

The orthogonal complement of a subspace

Let $V \subset \mathbb{F}_q^n$. The orthogonal complement of V is $V^\perp$ such that

$$V^\perp = \{\tilde{\mathbf{v}}_i \in \mathbb{F}_q^n \,|\, \langle \mathbf{v}_j, \tilde{\mathbf{v}}_i \rangle = 0, \text{ for all } \mathbf{v}_j \in V\}.$$

If V is m-dimensional, then $V^\perp$ is $(n-m)$-dimensional.

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→ Oil and Vinegar becomes Unbalanced Oil and Vinegar because of this attack.

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→ The attack can be generalised to find a vector in the intersection of more than two subspaces.