Algebraic cryptanalysis: MQ solving

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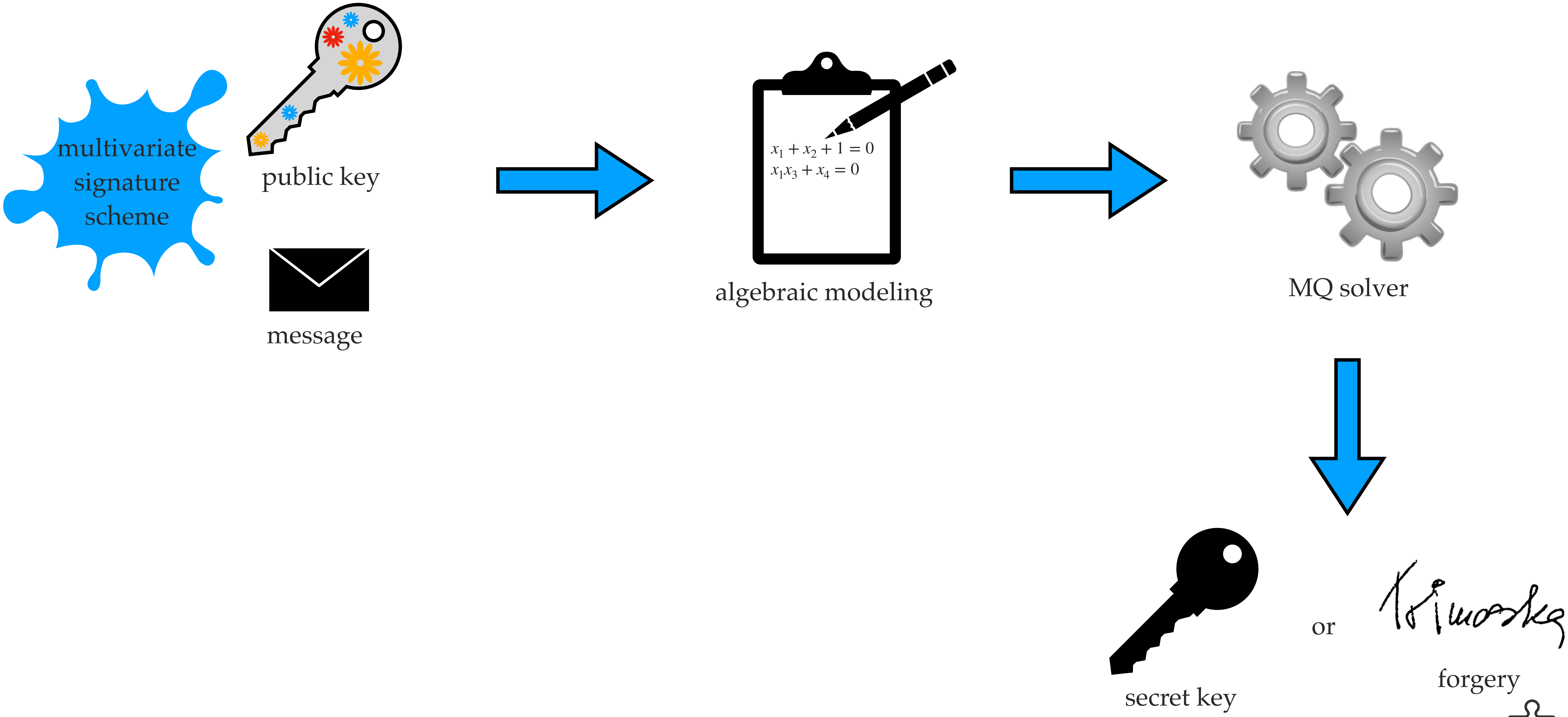
Selected Areas in Cryptology - Part 1
Spring, 2024

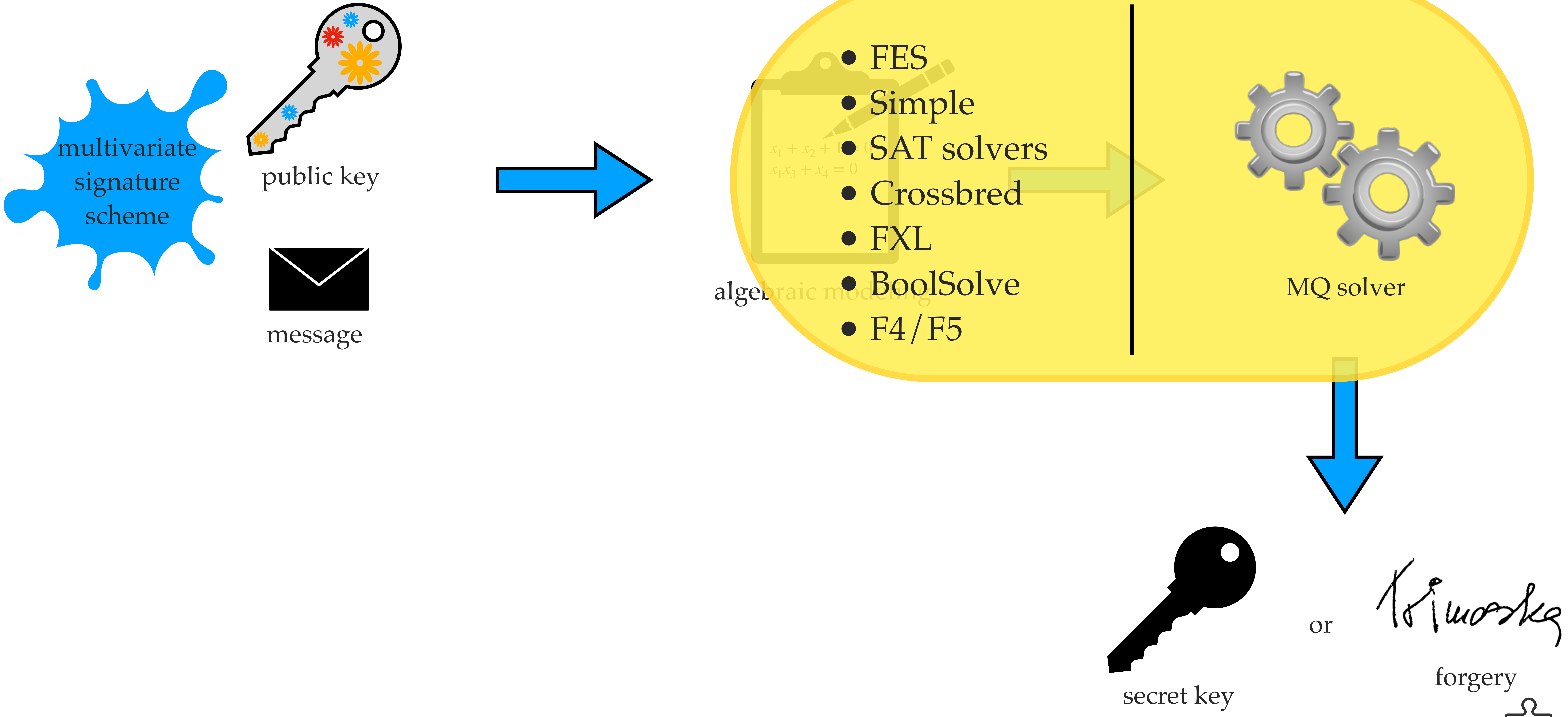
Algebraic cryptanalysis

A type of cryptanalytic methods where the problem of finding the secret key (or any attack goal) is reduced to the problem of finding a solution to a nonlinear multivariate polynomial system of equations.

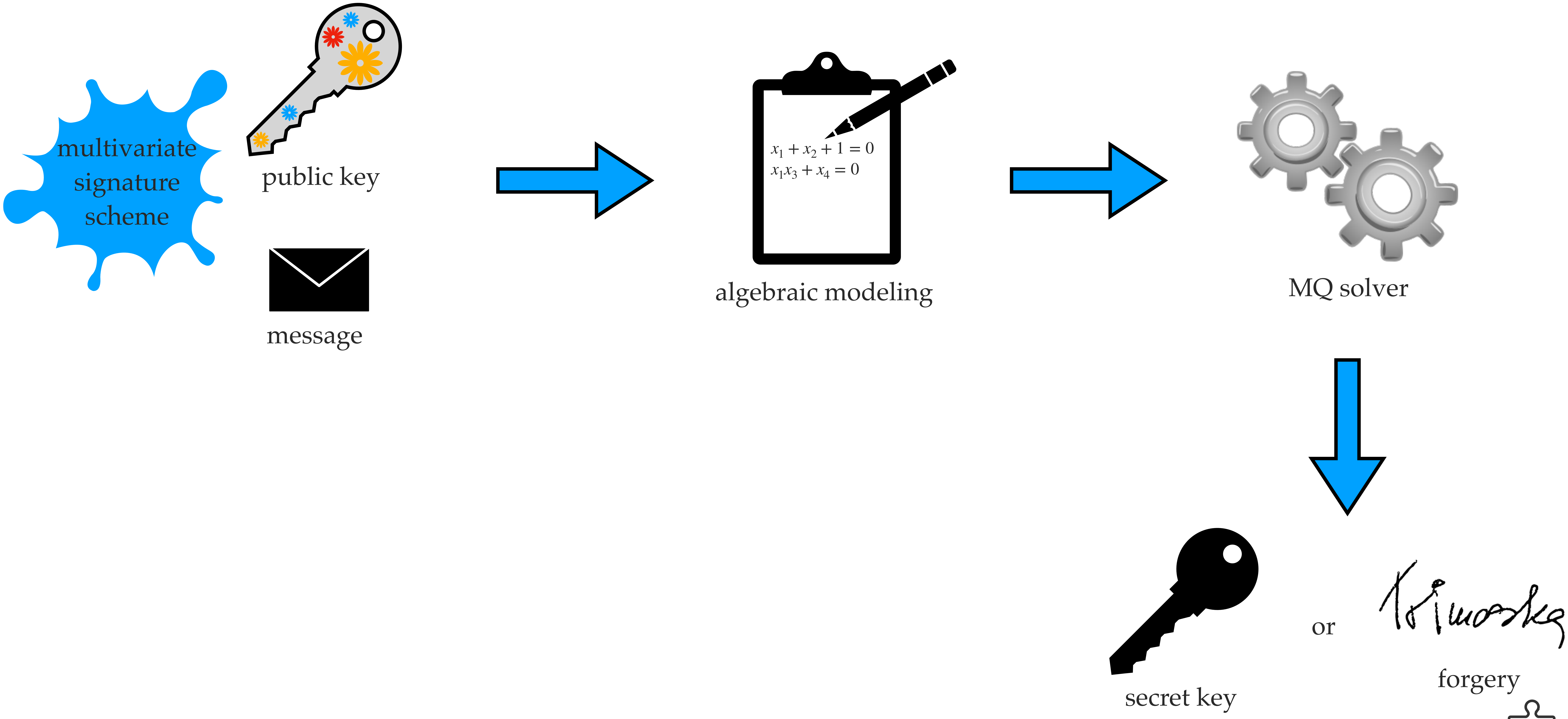
Algebraic cryptanalysis



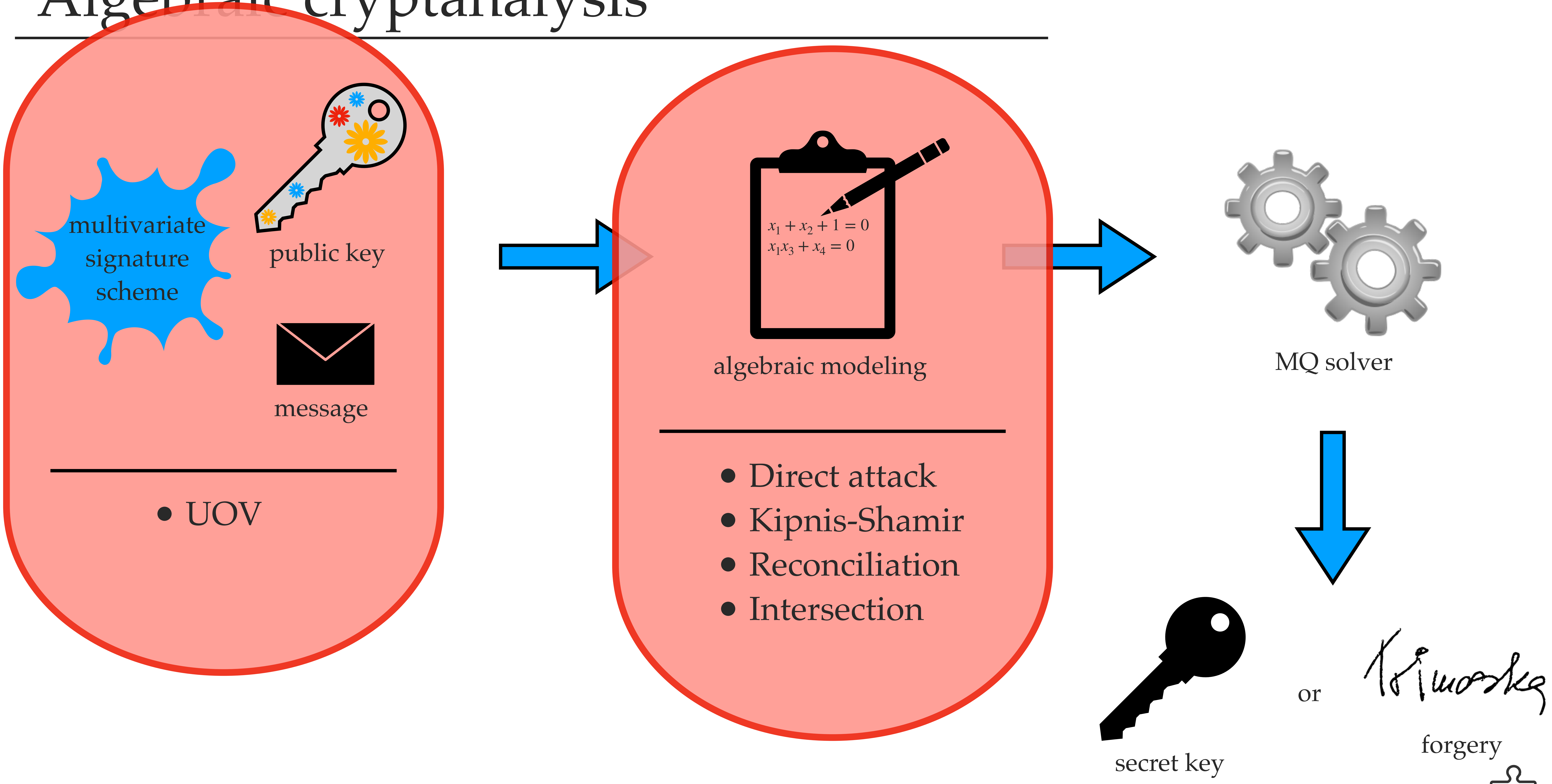
Algebraic cryptanalysis



Algebraic cryptanalysis



Algebraic cryptanalysis



The MQ problem

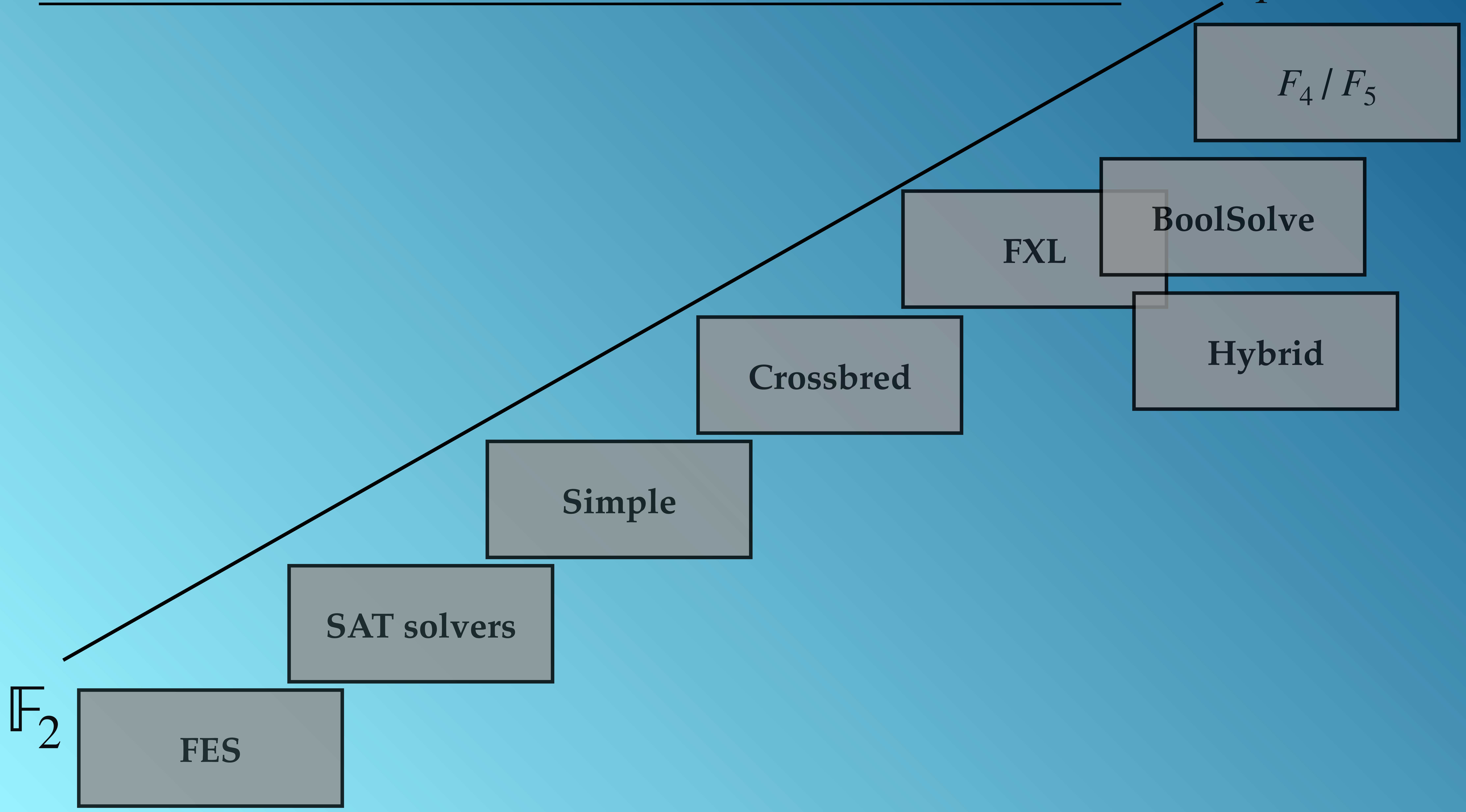
The MQ problem

Given m multivariate quadratic polynomials $f_1, \ldots, f_m$ of n variables over a finite field $\mathbb{F}_q$, find a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$, such that $f_1(\mathbf{x}) = \ldots = f_m(\mathbf{x}) = 0$.

Example.

$$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$$
$$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$$
$$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$$
$$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$$
$$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$$
$$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$$

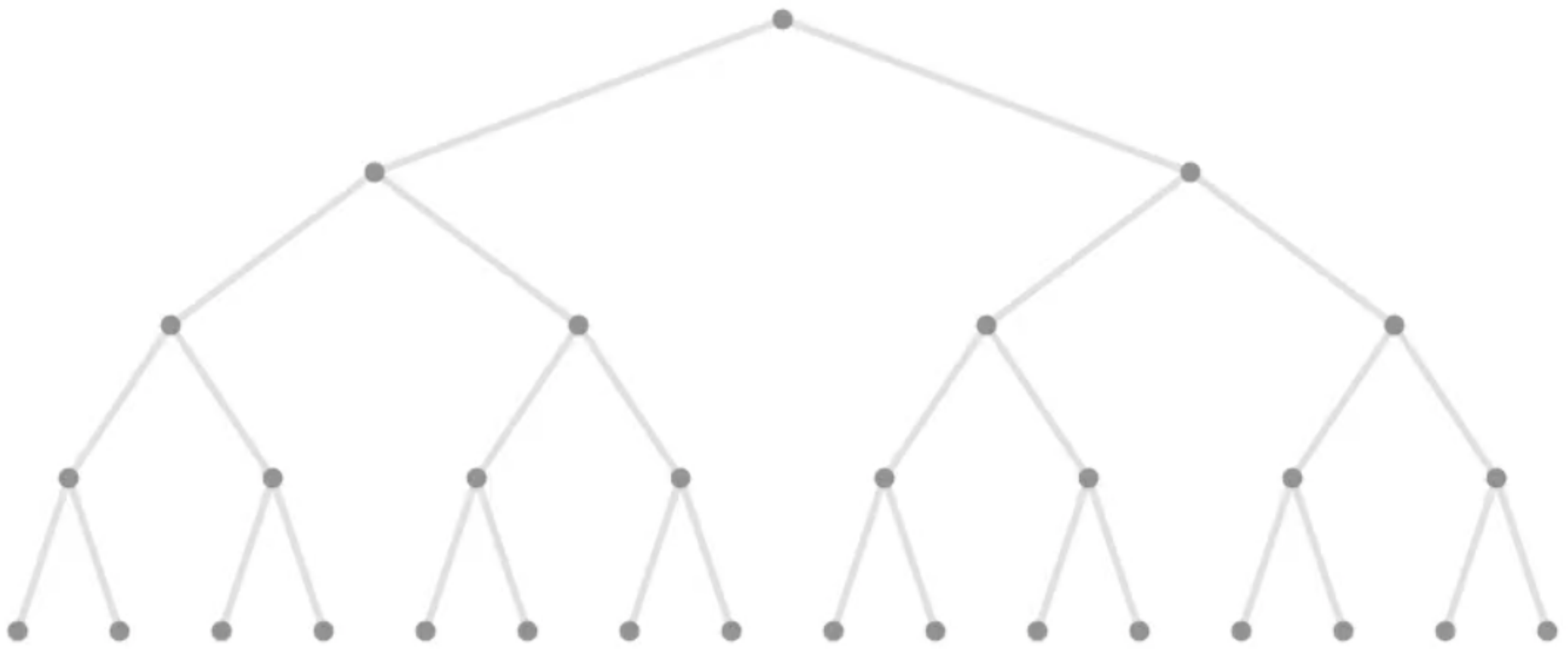
Overview of solvers



Techniques in

(Fast) Exhaustive Search

Exhaustive Search

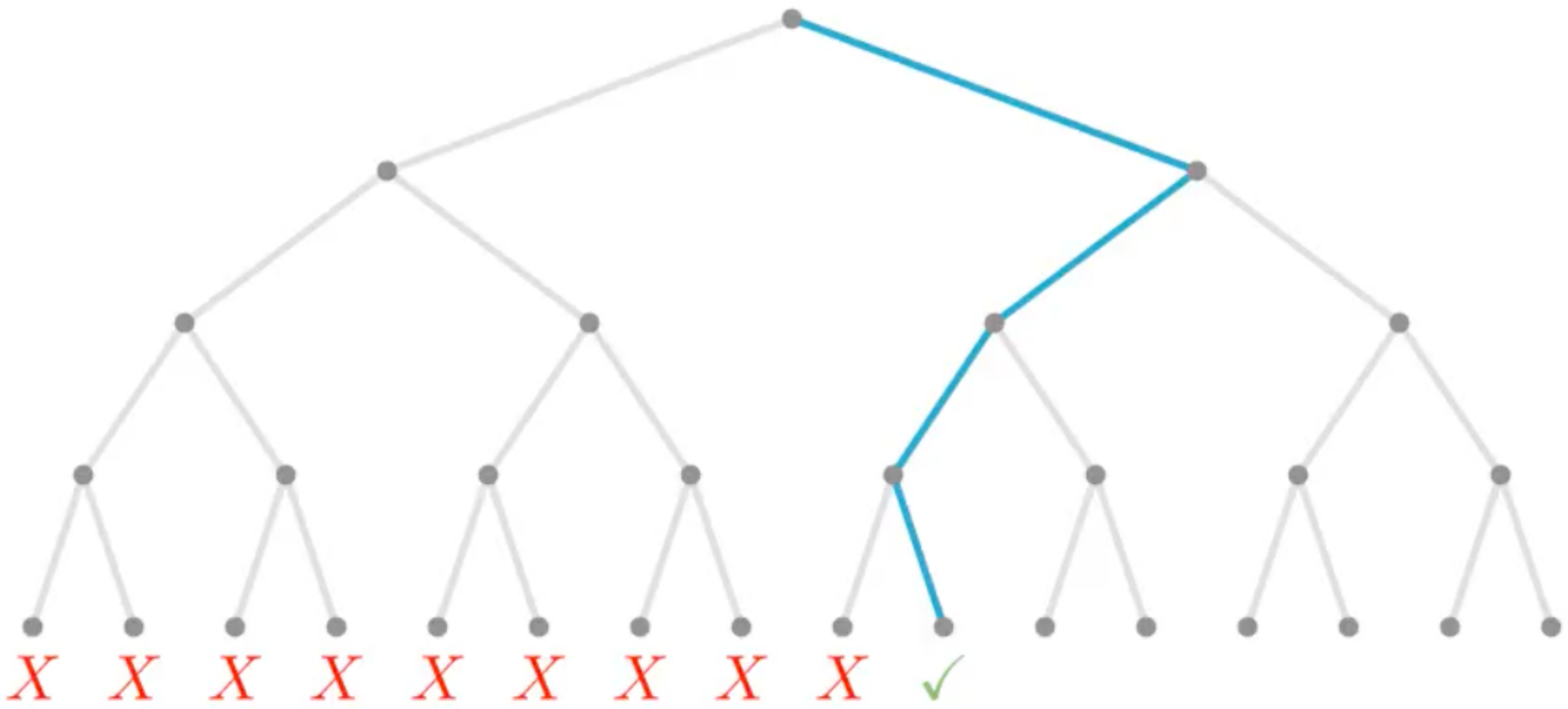


Binary search tree

$$x_1 \cdot x_2 + x_1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$
$$x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_2 + 1 = 0$$
$$x_1 \cdot x_2 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_4 = 0$$
$$x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 + x_3 + x_4 = 0$$

Exhaustive Search

Worst-case complexity: $\mathcal{O}(2^n)$



Binary search tree

$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$
$$0 \cdot 0 + 0 \cdot 1 + 1 + 0 + 1 = 0$$
$$1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 + 1 + 1 = 0$$
$$1 \cdot 1 + 0 \cdot 0 + 0 + 0 + 1 = 0$$

*Iterate example at: https://mtrimoska.com/SAC_MM/1_MQ_solving/

Fast Exhaustive Search

Gray code

- An ordering of the binary system where two successive values differ in only one bit.

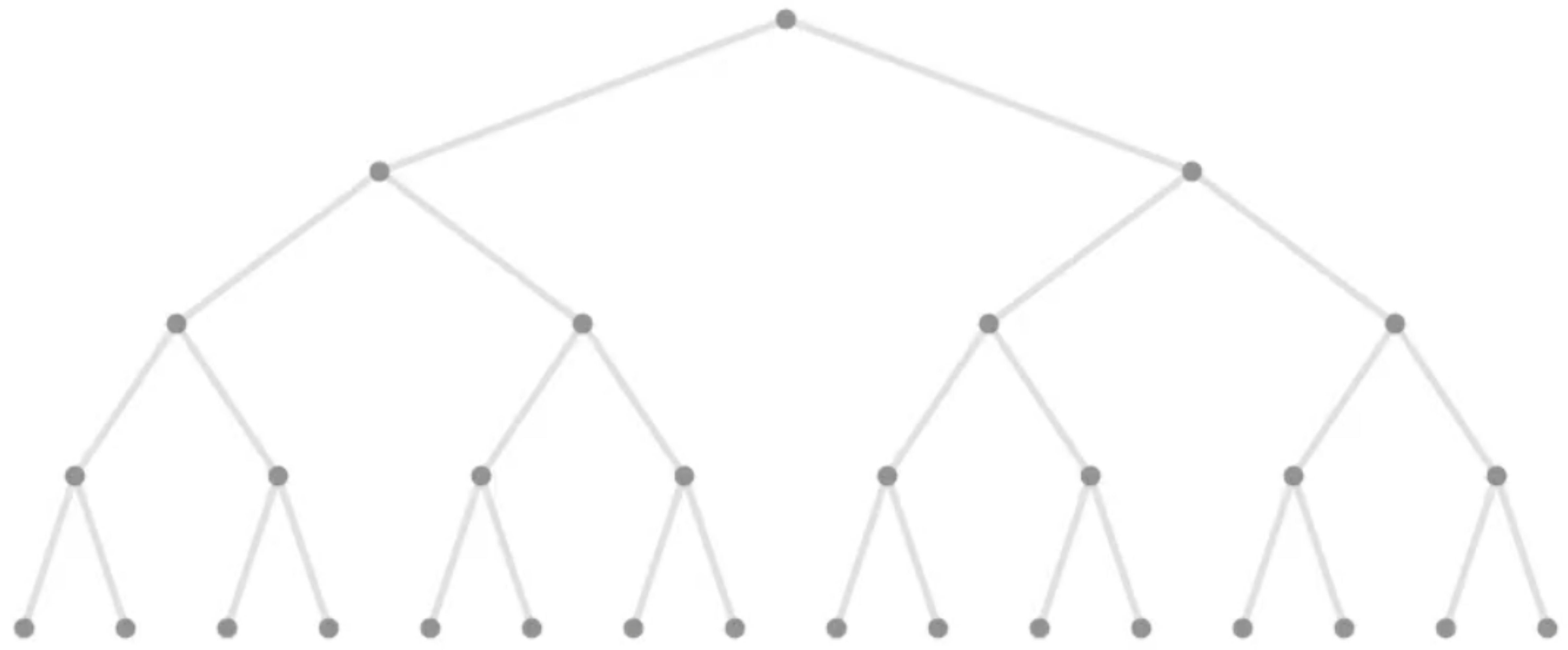
Example. $n = 4$

0000	1100
0001	1101
0011	1111
0010	1110
0110	1010
0111	1011
0101	1001
0100	1000

Fast Exhaustive Search

Gray code

0000	1100
0001	1101
0011	1111
0010	1110
0110	1010
0111	1011
0101	1001
0100	1000



$$x_1 \cdot x_2 + x_1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$

$$x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_2 + 1 = 0$$

$$x_1 \cdot x_2 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_4 = 0$$

$$x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 + x_3 + x_4 = 0$$

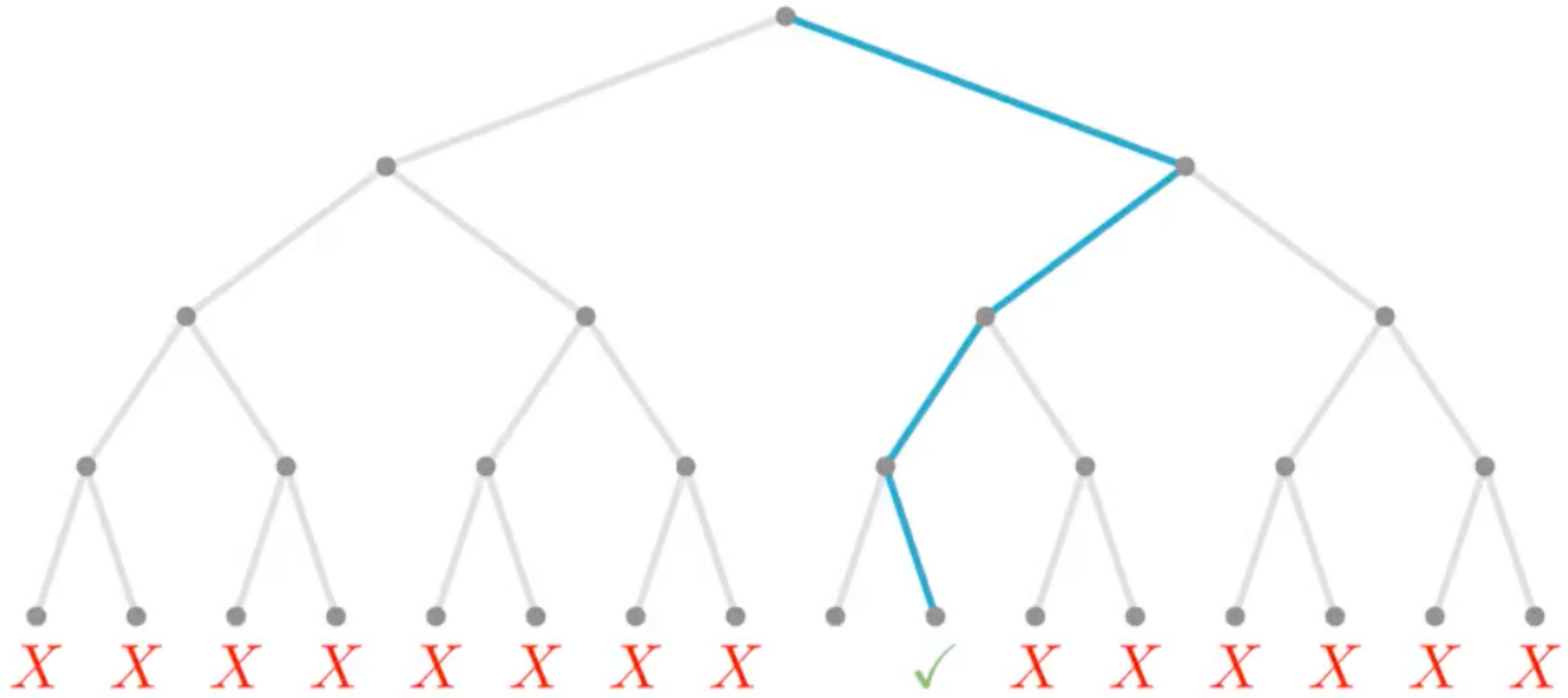
Fast Exhaustive Search

Worst-case complexity: $\mathcal{O}(2^n)$

! But, it differs from the depth-first traversal in the polynomial factors

Gray code

0000	1100
0001	1101
0011	1111
0010	1110
0110	1010
0111	1011
0101	1001
0100	1000



$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$
$$0 \cdot 0 + 0 \cdot 1 + 1 + 0 + 1 = 0$$
$$1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 + 1 + 1 = 0$$
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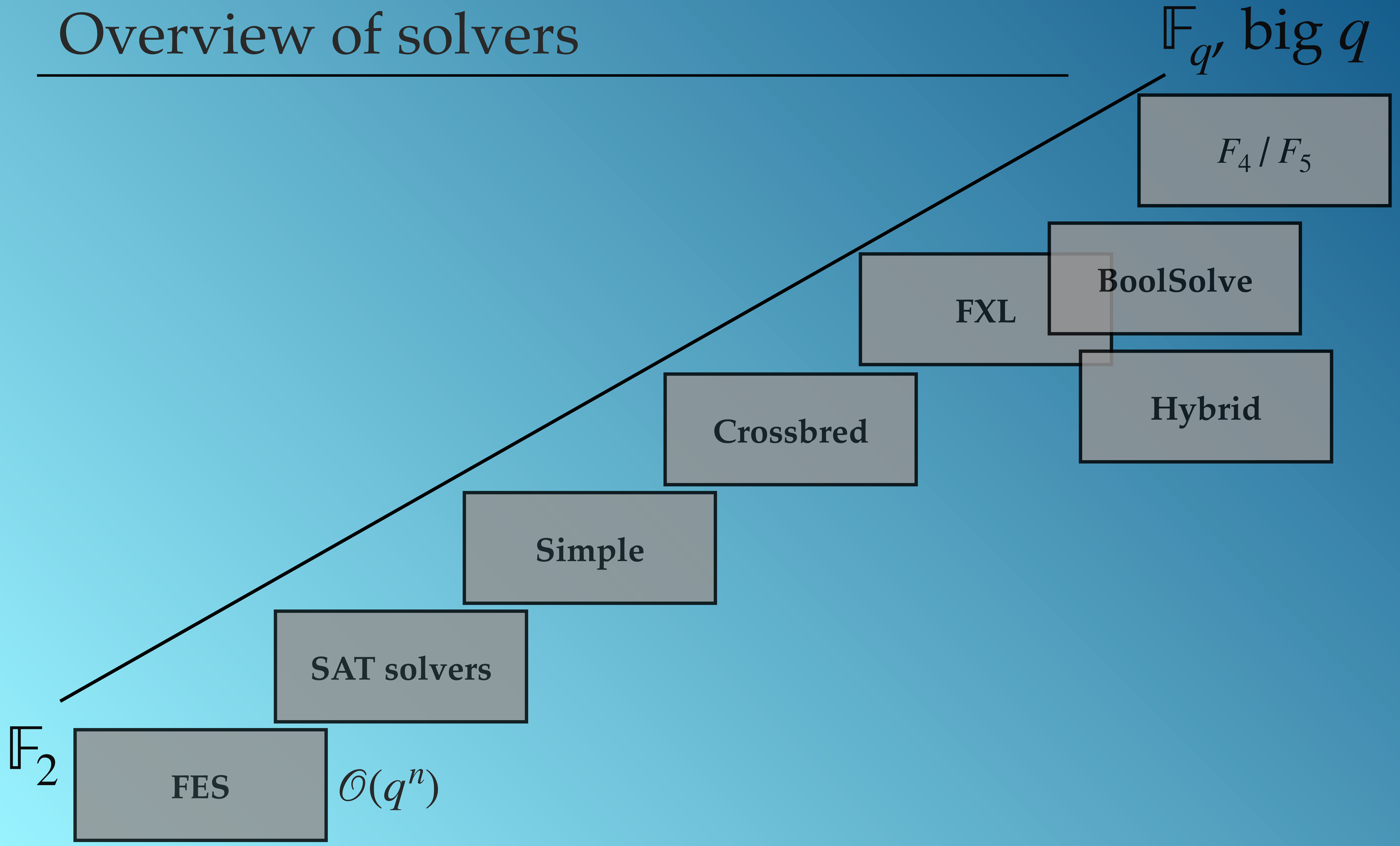
Overview of solvers



Techniques in

SAT solvers

(SAT solvers)

- Propositional formula in Conjunctive Normal Form (CNF): a conjunction of clauses where each clause is a disjunction of literals and where each literal is a variable or a negated variable.

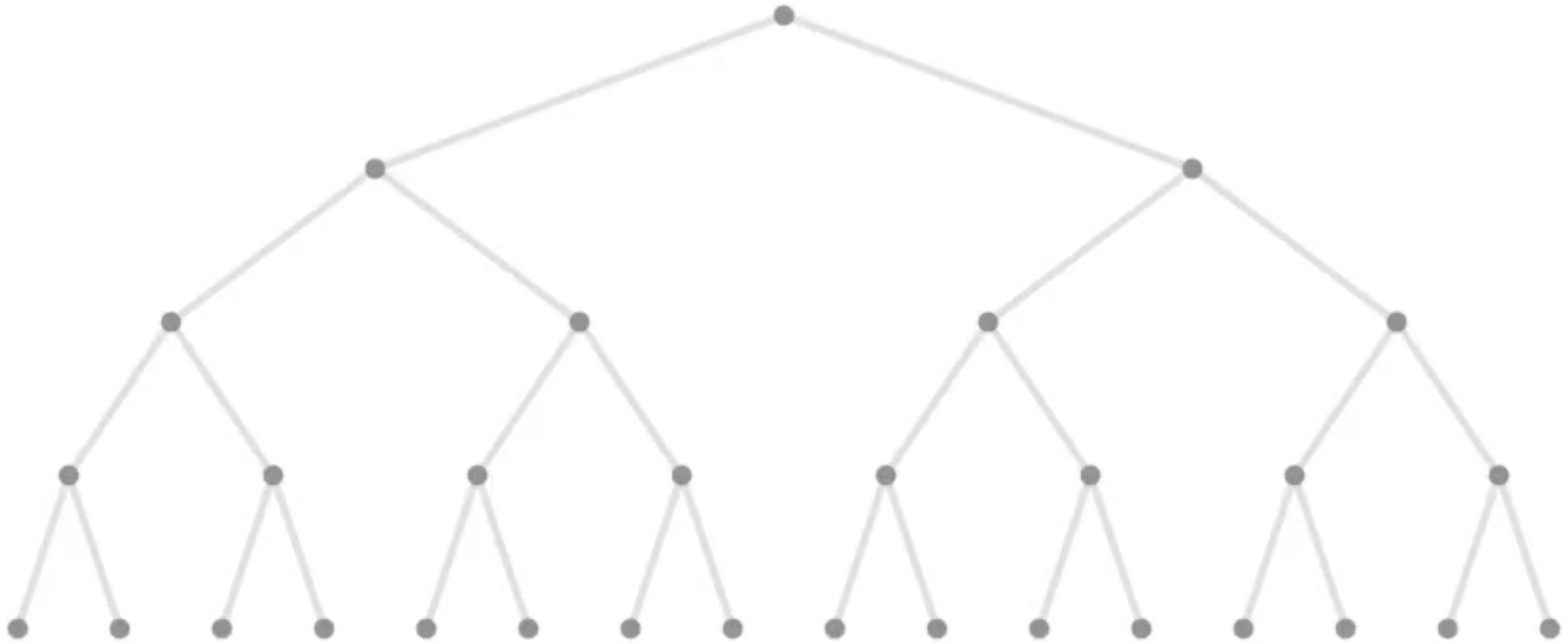
Example.
$$(x_1 \vee \neg x_2) \wedge$$
$$(x_2 \vee x_3 \vee x_4) \wedge$$
$$(\neg x_1 \vee x_4)$$

The SATisfiability problem

Given a propositional formula, determine whether there exists an interpretation (assignment of all variables) such that the formula is satisfied (evaluates to TRUE).

SAT solver: a tool for solving the SAT problem.

Partial assignment and conflicts



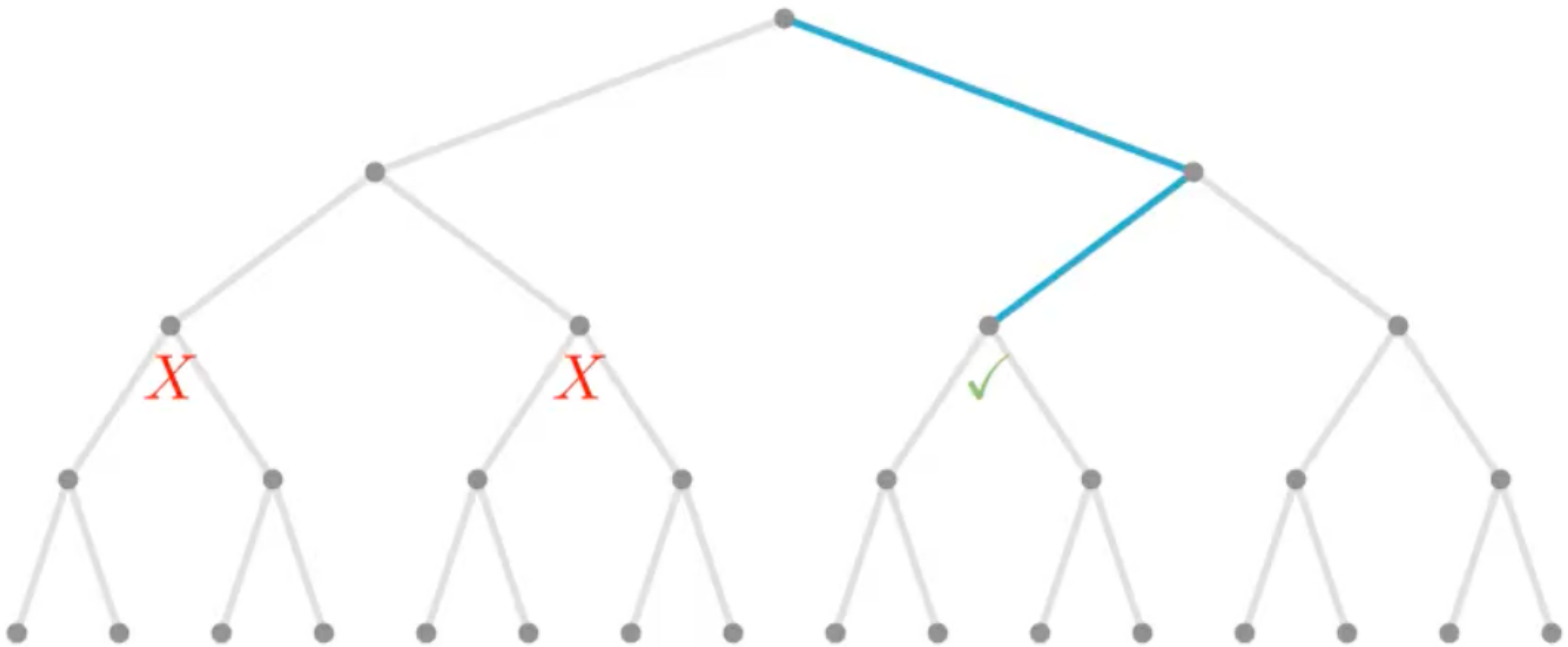
$$x_1 \cdot x_2 + x_1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$

$$x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_2 + 1 = 0$$

$$x_1 \cdot x_2 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_4 = 0$$

$$x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 + x_3 + x_4 = 0$$

Partial assignment and conflicts



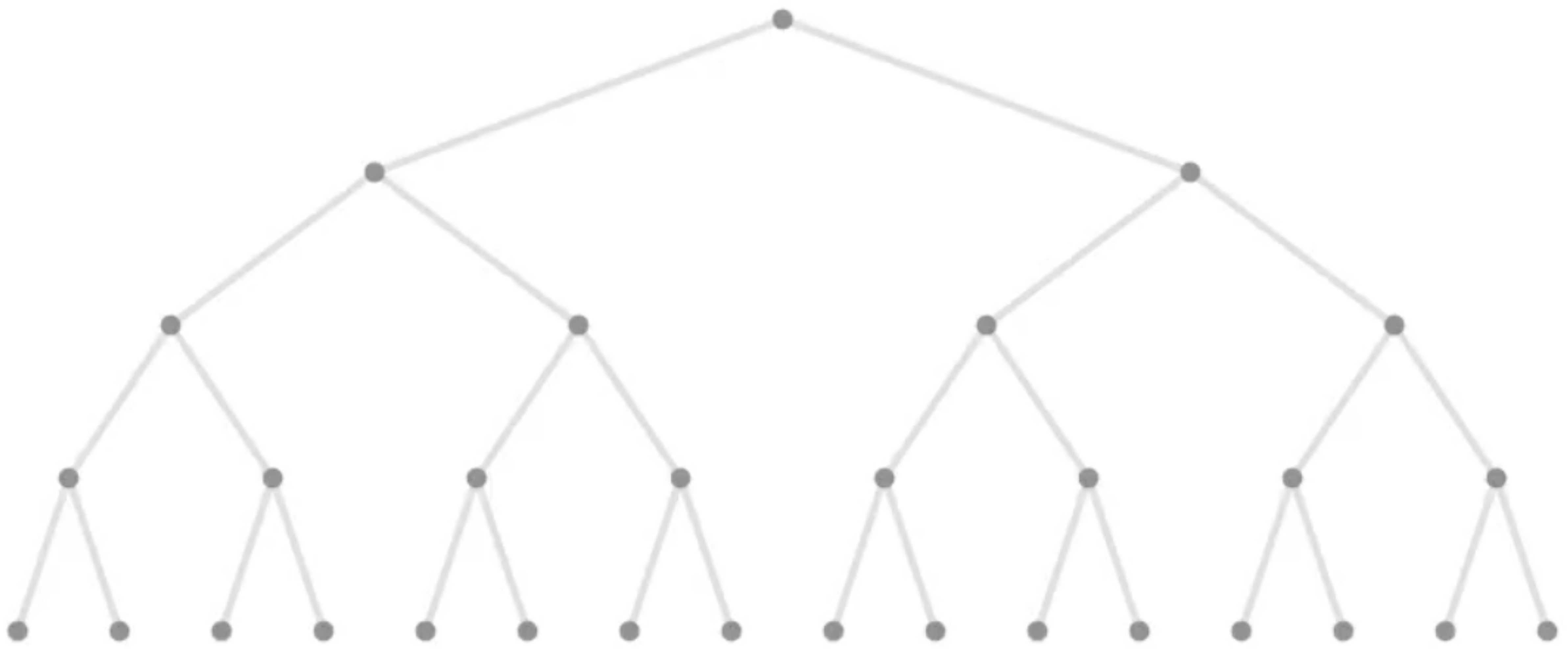
$$1 \cdot 0 + 1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$
$$0 \cdot x_3 + 0 \cdot x_4 + 1 + 0 + 1 = 0$$
$$1 \cdot 0 + 0 \cdot x_3 + 0 \cdot x_4 + 1 + x_4 = 0$$
$$1 \cdot x_4 + 0 \cdot x_3 + 0 + x_3 + x_4 = 0$$

*Iterate example at: https://mtrimoska.com/SAC_MM/1_MQ_solving/

Partial assignment and conflicts

Which (portion of) branches are missing ??

Worst-case complexity: $\mathcal{O}(2^n)$



$$x_1 \cdot x_2 + x_1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$

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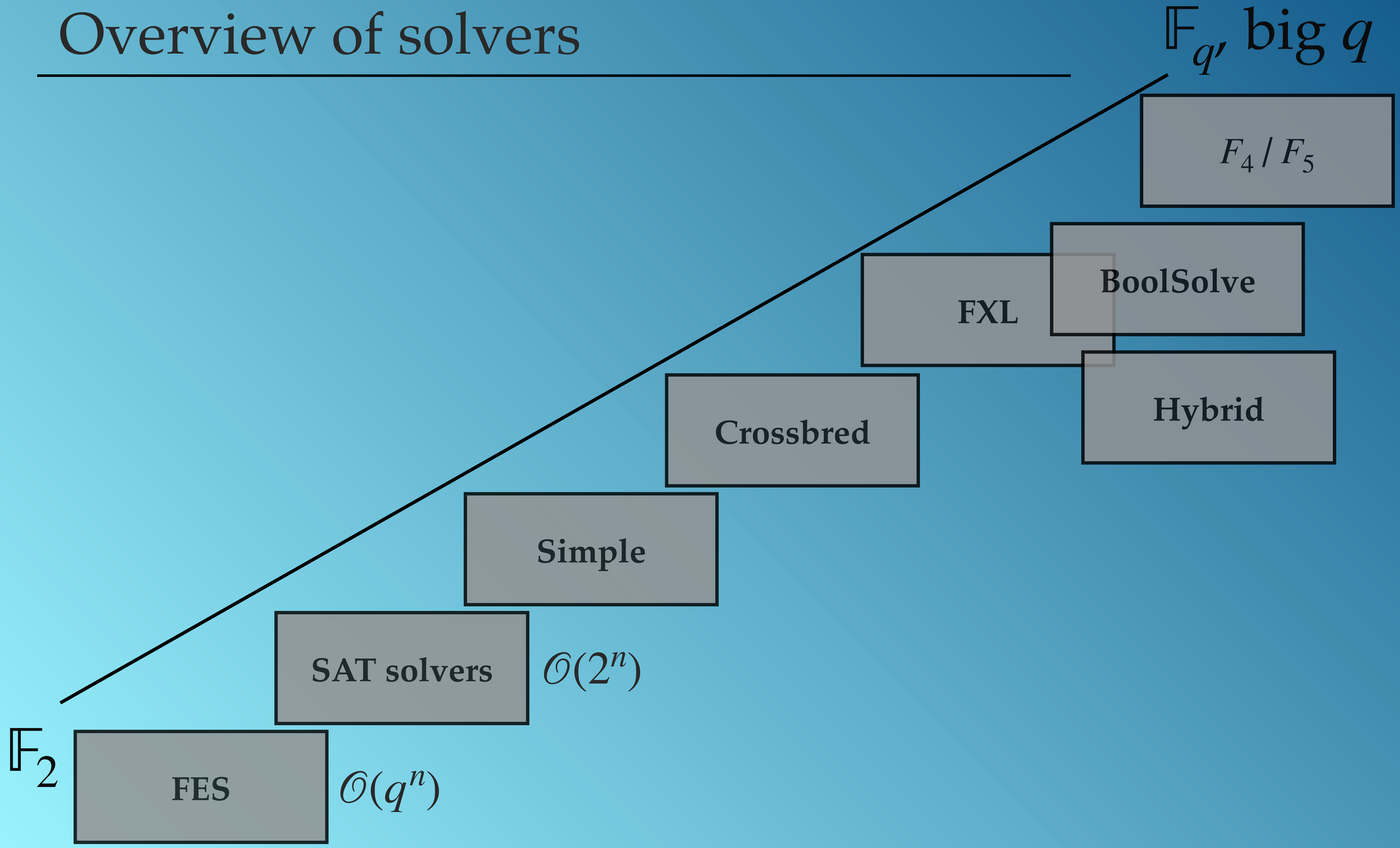
$$x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 + x_3 + x_4 = 0$$

XOR-enabled SAT solvers: take as input XOR constraints as well; perform Gaussian elimination;
*CryptoMiniSat, WDSat

Overview of solvers



Macaulay matrix

Linearisation

Linear systems are easy to solve, nonlinear systems are hard.

Linearisation: for each nonlinear monomial, replace all of its occurrences by a new variable.

Example.

$$
\begin{aligned}
f_1 &: x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0\\
f_2 &: x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0\\
f_3 &: x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0\\
f_4 &: \boxed{x_1x_2} + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0\\
f_5 &: \boxed{x_1x_2} + x_2x_3 + x_1x_4 + x_3 = 0\\
f_6 &: x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
\quad\longrightarrow\quad
\begin{aligned}
f_1 &: y_2 + y_5 + x_1 + x_3 + x_4 = 0\\
f_2 &: y_4 + y_3 + y_6 + x_1 + x_2 + x_4 = 0\\
f_3 &: y_5 + y_6 + x_1 + x_3 + 1 = 0\\
f_4 &: \boxed{y_1} + y_2 + y_4 + x_3 + x_4 + 1 = 0\\
f_5 &: \boxed{y_1} + y_4 + y_3 + x_3 = 0\\
f_6 &: y_2 + y_3 + y_6 + x_1 + x_2 + x_3 + x_4 = 0
\end{aligned}
$$

Linearisation

<u>Linearisation adds solutions</u>: a *random* quadratic system of m equations in n variables, when $n = m$, is expected to have one solution (probability is $\sim \frac{1}{q}$ for systems over $\mathbb{F}_q$). The corresponding linearised system has a solution space of dimension $\binom{n+1}{2} - m$.

$\binom{n}{2}$ quadratic plus n linear monomials

<u>Loss of information</u>: e.g. assignment $x_1 = 1; x_2 = 0; y_1 = 1;$ is part of a valid solution to the linearised system, but $x_1 x_2 \neq y_1$.

Macaulay matrix

Monomials →

Equations ↓

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$

$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$

$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$

$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$

$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$

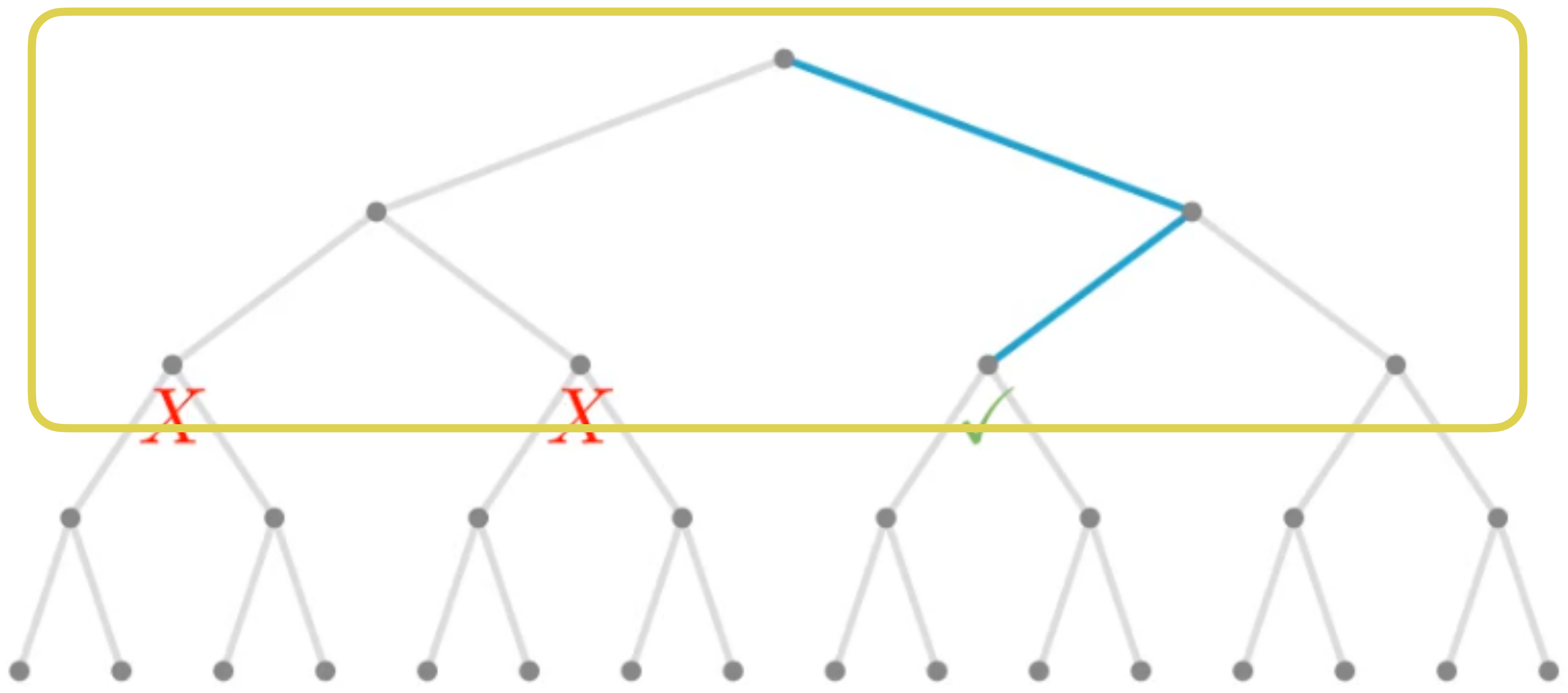
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

Techniques in

Simple algorithm

Simple algorithm

- Partial assignment
  - Gaussian elimination



$$1 \cdot 0 + 1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$
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$$1 \cdot 0 + 0 \cdot x_3 + 0 \cdot x_4 + 1 + x_4 = 0$$
$$1 \cdot x_4 + 0 \cdot x_3 + 0 + x_3 + x_4 = 0$$

Simple algorithm

Guess sufficiently many variables so that the remaining polynomial system can be solved by linearization.

Simple algorithm: complexity

- n - number of variables
- m - number of equations

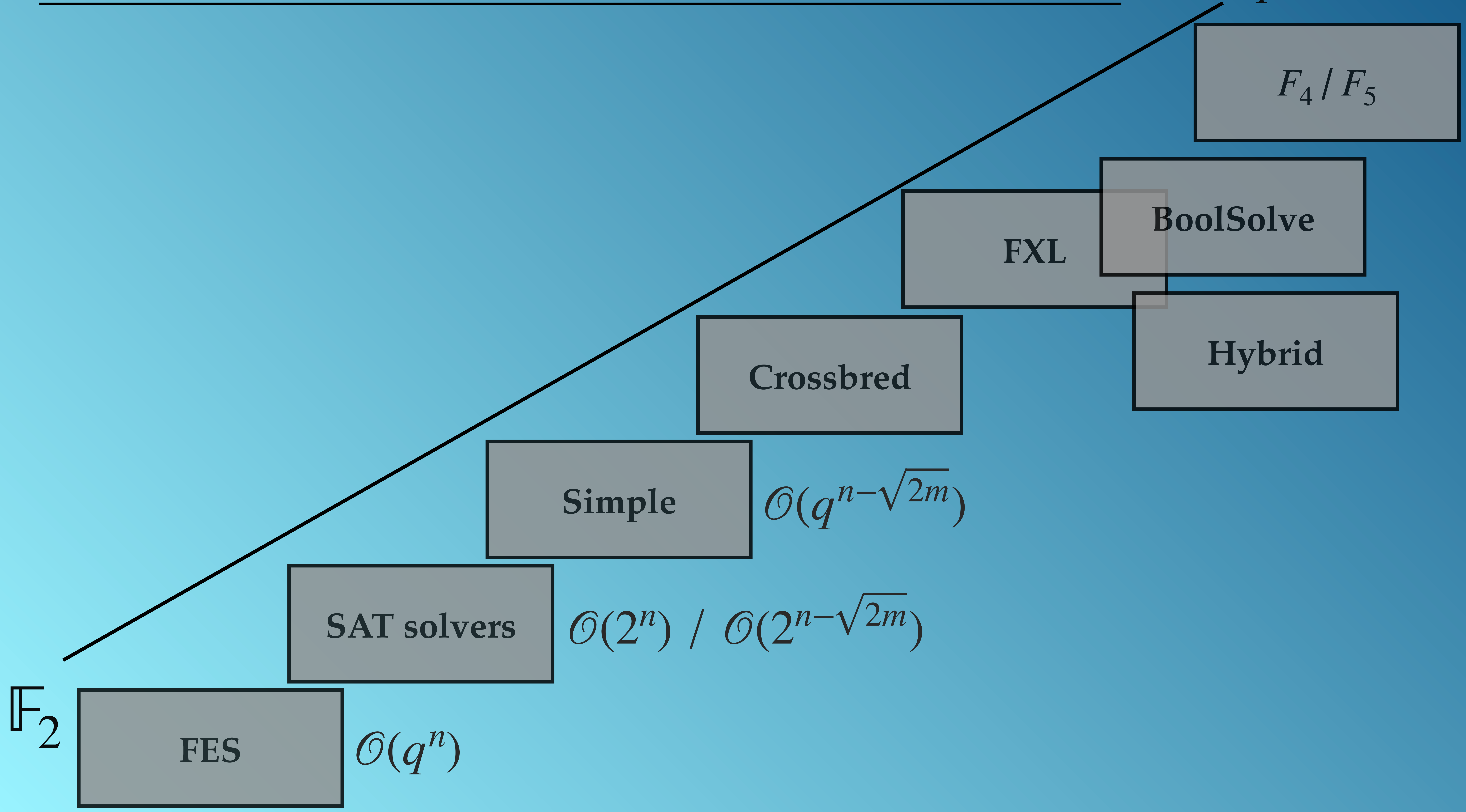
Enumeration ends when:

number of monomials $\leq$ number of equations

$$\binom{n-?}{2} \leq m$$

$$\mathcal{O}(2^{n-\sqrt{2m}})$$

Overview of solvers



Techniques in

Gröbner basis algorithms

Gröbner basis algorithms (intuition)

*We are essentially describing the XL algorithm.

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

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*We are essentially describing the XL algorithm.

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
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$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

$\boxed{D = 3}$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0				
f_2	0	0	1	1	1	0	1	1	0	1	0				
f_3	0	0	0	1	0	1	0	1	1	0	1				
f_4	1	1	0	1	1	0	0	0	1	1	1				
f_5	1	0	1	1	1	0	0	0	1	0	0				
f_6	0	1	1	1	0	0	1	1	1	1	0				
x_1f_1															
x_2f_1															
$\dots$															

Gröbner basis algorithms (intuition)

*We are essentially describing the XL algorithm.

$\boxed{D = 4}$

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$	$x_1x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0					
f_2	0	0	1	1	1	0	1	1	0	1	0					
f_3	0	0	0	1	0	1	0	1	1	0	1					
f_4	1	1	0	1	1	0	0	0	1	1	1					
f_5	1	0	1	1	1	0	0	0	1	0	0					
f_6	0	1	1	1	0	0	1	1	1	1	0					
x_1f_1																
x_2f_1																
$\cdots$																
$x_1x_2f_1$																
$x_1x_3f_1$																

Gröbner basis

- Let $R = \mathbb{F}_q[x_1, \ldots, x_n]$ be the polynomial ring in n variables.

- An ideal in R is an additive subgroup I such that if $g \in R$ and $f \in I$, then $gf \in I$.

- The subset $\{f_1, \ldots, f_m\} \subset R$ is a set of generators for an ideal I if every element $t \in I$ can be written in the form $t = \sum_{1}^{n}$ with $g_i \in R$.

- By the Hilbert basis theorem: every ideal in R has a finite set of generators.

- The subset of R defined as $V(I) = \{(a_1, \ldots, a_n) \in \mathbb{F}_q^n \,|\, f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}$ is called an algebraic variety. It is the set of all solutions to the system of equations $f_1(x_1, \ldots, x_n) = \ldots = f_1(x_1, \ldots, x_n) = 0$.

- By the Nullstellensatz: $\mathbf{I}(V(I)) = I$, where $\mathbf{I}(V)$ denotes the ideal of V, i.e. $\mathbf{I}(V) = \{f \in R \,|\, f(a) = 0 \text{ for all } a \in V\}$ (Similar to Gauss' fundamental theorem, but for polynomials in many variables).

Gröbner basis

- A Gröbner basis of an ideal I is a set of generators with some nice (useful) property.

For our case, the nice property is that a solution can be extracted easily from the Gröbner basis.

Example. The shape of a GB with respect to the lexicographic order

$$
\begin{aligned}
f_1 &: x_1x_3 + x_1 + x_2x_4 + x_5 + x_6 + 1 = 0\\
f_2 &: x_1x_4 + x_1 + x_2x_3 + x_2 + x_3x_4 + x_3x_6 + x_4 + x_5 = 0\\
f_3 &: x_1x_5 + x_1 + x_2 + x_3x_4 + x_6 + 1 = 0\\
f_4 &: x_1x_2 + x_1x_3 + x_2x_5 + x_3 + x_4 + x_6 + 1 = 0\\
f_5 &: x_1x_4 + x_2x_3 + x_2x_5 + x_5x_6 + 1 = 0\\
f_6 &: x_1x_3 + x_1x_4 + x_1 + x_2 + x_3x_6 + x_3 + x_5 = 0
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
f'_1 &: x_1 + x_6 = 0\\
f'_2 &: x_2 + x_6 = 0\\
f'_3 &: x_3 + x_6 = 0\\
f'_4 &: x_4 + x_6 + 1 = 0\\
f'_5 &: x_5 = 0
\end{aligned}
$$

*****
****
***
**
*

$$V(< f_1, \ldots, f_6 >) = \{(0,0,0,1,0,0), (1,1,1,0,0,1)\}$$

Gröbner basis algorithms:

Buchberger, Lazard, F4, F5

Follow the core idea that we described, but combine the equations in an organised way, rather than multiplying them by all possible monomials.

Not covered in this course:

- Monomial orders
- S-polynomials
- Polynomial long division
- Row reduction in parallel
- Reductions to zero
- Syzygy criterion
- …

XL/Gröbner basis algorithms: complexity

$$\mathcal{O}\left(mD_{reg} \binom{n + D_{reg} - 1}{D_{reg}}^{\omega} \right)$$

D_{reg}: degree of regularity

the power of the first non-positive coefficient in the expansion of $$\frac{(1-t^2)^m}{(1-t)^n}$$

Overview of solvers

$\mathbb{F}_q$, big q

F_4 / F_5 $\mathcal{O}(\binom{n+D_{reg}-1}{D_{reg}}^{\omega})$

BoolSolve

FXL

Hybrid

Crossbred

Simple $\mathcal{O}(2^{n-\sqrt{2m}})$

SAT solvers $\mathcal{O}(2^n)$ / $\mathcal{O}(2^{n-\sqrt{2m}})$

$\mathbb{F}_2$

FES $\mathcal{O}(q^n)$

Techniques in

FXL, Hybrid, BoolSolve

FXL, Hybrid, BoolSolve

Techniques are already covered in the previous section.

Algorithms will be explained in the summary.

Techniques in

The crossbred algorithm

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1
f_1	0	1	0	1	0	1	0	0	1	1	0
f_2	0	0	1	1	1	0	1	1	0	1	0
f_3	0	0	0	1	0	1	0	1	1	0	1
f_4	1	1	0	1	1	0	0	0	1	1	1
f_5	1	0	1	1	1	0	0	0	1	0	0
f_6	0	1	1	1	0	0	1	1	1	1	0

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Put matrix in reduced row echelon form

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Take linear subsystem

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

…if we had another 4 equations

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

- Subsystem is linear in variables $\{x_1, x_2, x_3\}$.
  - Enumerating x_4 will result in a linear subsystem.

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1

…

Crossbred algorithm

$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$
$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$
$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$
$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$
$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$
$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$

→ Subsystem can be linearised

	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1
…											

…if we had another 4 equations, the subsystem would have a unique solution.

Otherwise: check candidate solutions against the other equations.

Crossbred algorithm

Parameters of the algorithm: D, k, d, h

- Enumerate h variables.
- Choose k of the remaining variables.
- Augment system up to degree D (compute degree-D Macaulay matrix).
- Take the subsystem that is at most degree d in the k chosen variables.
- Enumerate all but the k chosen variables.
- Linearise the subsystem and solve it.
- Check if candidate solutions are consistent with the rest of the system.

The complexity is calculated as the best trade-off between the four parameters.

Crossbred algorithm

	Number of Variables (n)	Seed (0,1,2,3,4)	Date	Contestants	Computational Resource	Data
1	83	0	2023/09/16	Charles Bouillaguet and Julia Sauvage	https://gitlab.lip6.fr/almasty/hpXbred, 3488 AMD EPYC 7J13 cores on the Oracle public cloud	Details
6	74	0	2016/12/17	Antoine Joux	New hybridized XL related algorithm, Heterogeneous cluster of Intel Xeon @ 2.7-3.5 Ghz	Details
7	74	4	2017/11/15	Kai-Chun Ning, Ruben Niederhagen	Parallel Crossbred, 54 GPUs in the Saber cluster	Details
25	66	0	2016/01/22	Tung Chou, Ruben Niederhagen, Bo-Yin Yang	Gray Code enumeration, Rivyera, 128 Spartan 6 FPGAs	Details

Fukuoka MQ challenge record computations ($m = 2n$)

Overview of solvers

$\mathbb{F}_q$, big q

F_4 / F_5 $\mathcal{O}(\binom{n+D_{reg}-1}{D_{reg}}^{\omega})$

BoolSolve

FXL

Hybrid

Crossbred $\mathcal{O}(\ldots)$

Simple $\mathcal{O}(2^{n-\sqrt{2m}})$

SAT solvers $\mathcal{O}(2^n)$ / $\mathcal{O}(2^{n-\sqrt{2m}})$

$\mathbb{F}_2$

FES $\mathcal{O}(q^n)$

Summary

(Partial) enumeration

Candidate solutions (subsystem)

Conflict search

Extending to higher degrees

Computing a Gröbner Basis

FES

Simple

FXL

F_4 / F_5

SAT solvers

Crossbred

BoolSolve

Hybrid

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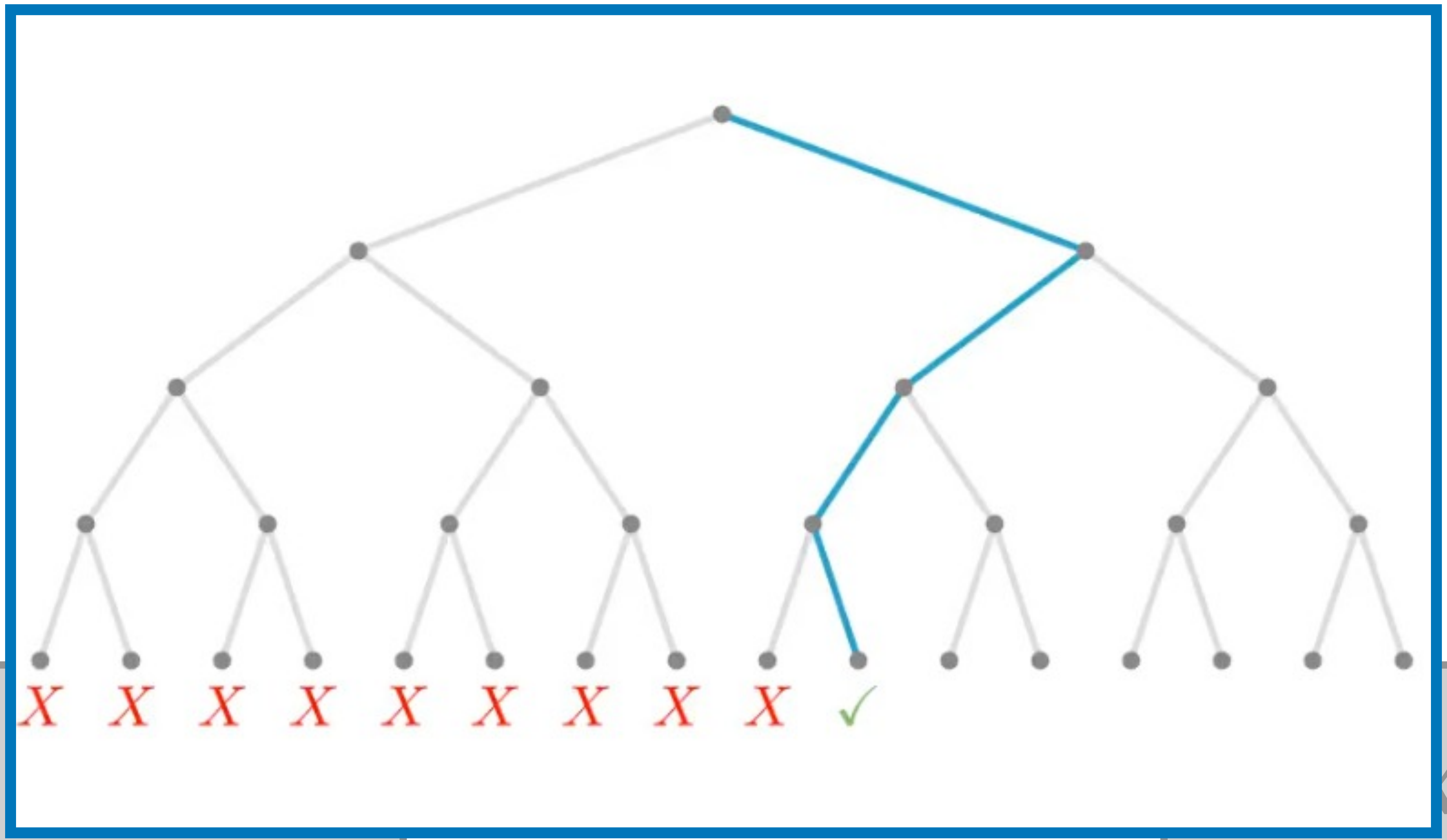

FES

XL

F_4 / F_5

SAT solvers

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	x_1x_2	x_1x_3	x_2x_3	x_1x_4	x_2x_4	x_3x_4	x_1	x_2	x_3	x_4	1
f_1	1	0	0	0	0	0	0	0	0	1	1
f_2	0	1	0	0	0	0	1	1	1	1	0
f_3	0	0	1	0	0	0	1	1	0	1	0
f_4	0	0	0	1	0	0	1	1	1	0	1
f_5	0	0	0	0	1	0	0	1	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	1
$\dots$											

FES

F_4 / F_5

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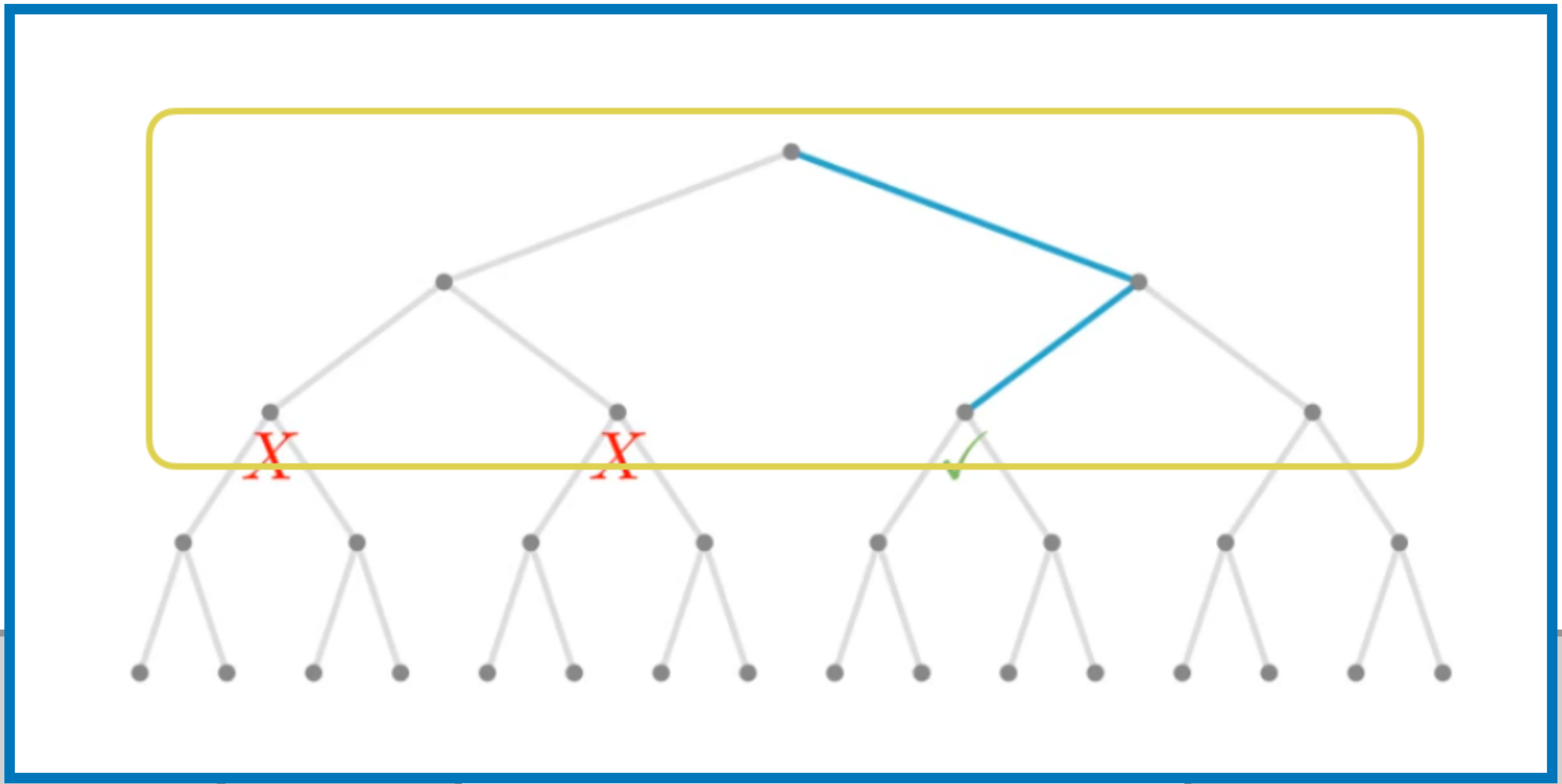

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	x_1x_2	x_1x_3	x_1x_4	x_1	x_2x_3	x_2x_4	x_2	x_3x_4	x_3	x_4	1	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$	$x_1x_2x_3x_4$
f_1	0	1	0	1	0	1	0	0	1	1	0					
f_2	0	0	1	1	1	0	1	1	0	1	0					
f_3	0	0	0	1	0	1	0	1	1	0	1					
f_4	1	1	0	1	1	0	0	0	1	1	1					
f_5	1	0	1	1	1	0	0	0	1	0	0					
f_6	0	1	1	1	0	0	1	1	1	1	0					
x_1f_1																
x_2f_1																
$\cdots$																
$x_1x_2f_1$																
$x_1x_3f_1$																

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$$f_1' : x_1 + x_6 = 0$$
$$f_2' : x_2 + x_6 = 0$$
$$f_3' : x_3 + x_6 = 0$$
$$f_4' : x_4 + x_6 + 1 = 0$$
$$f_5' : x_5 = 0$$

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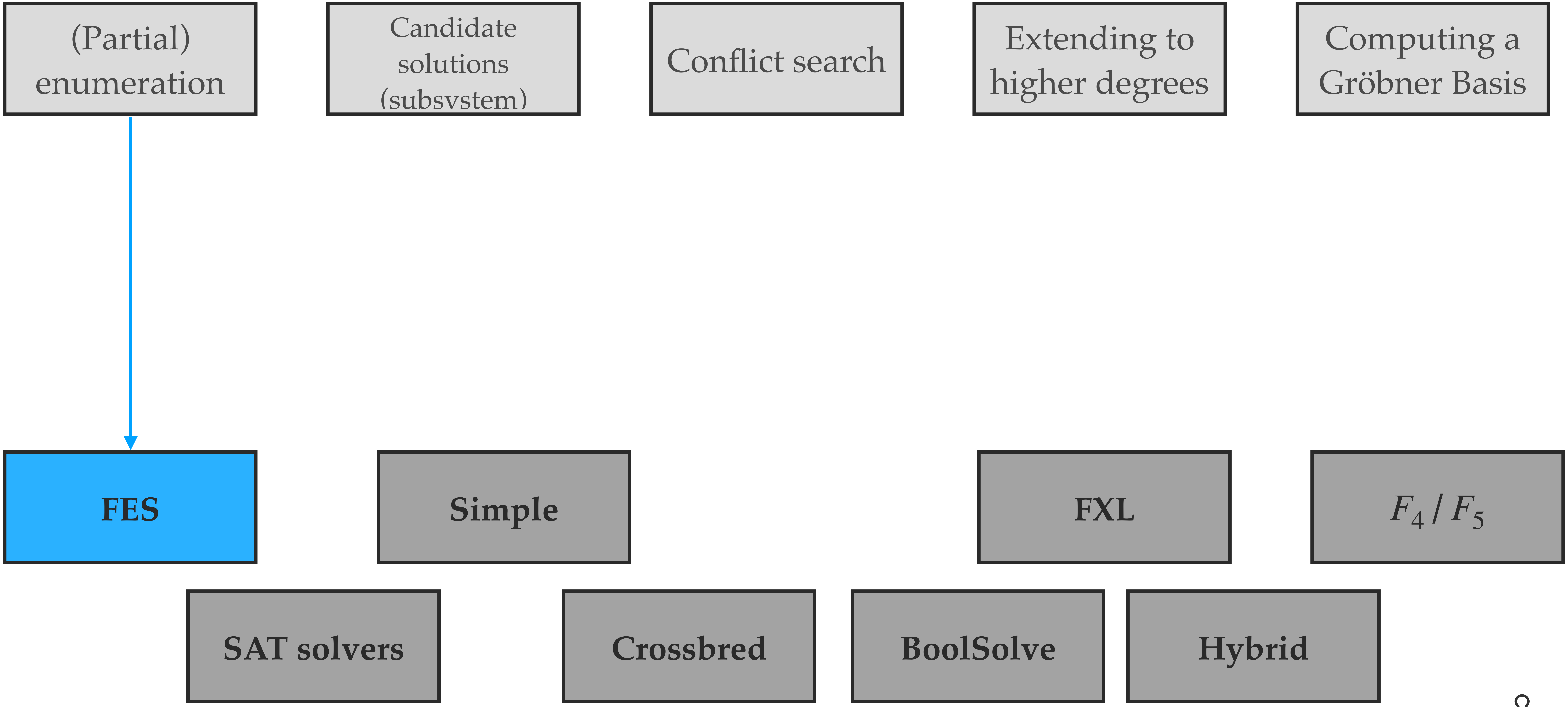
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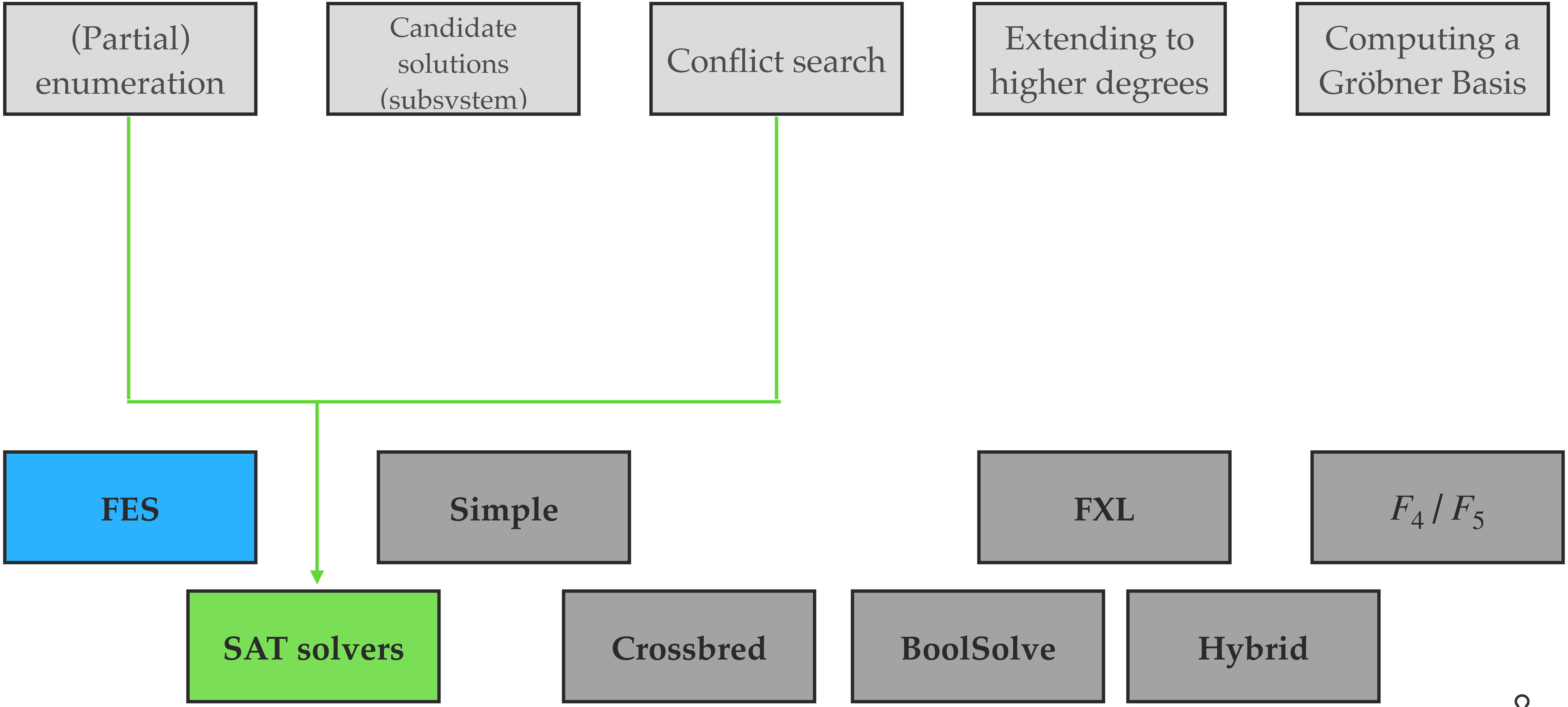
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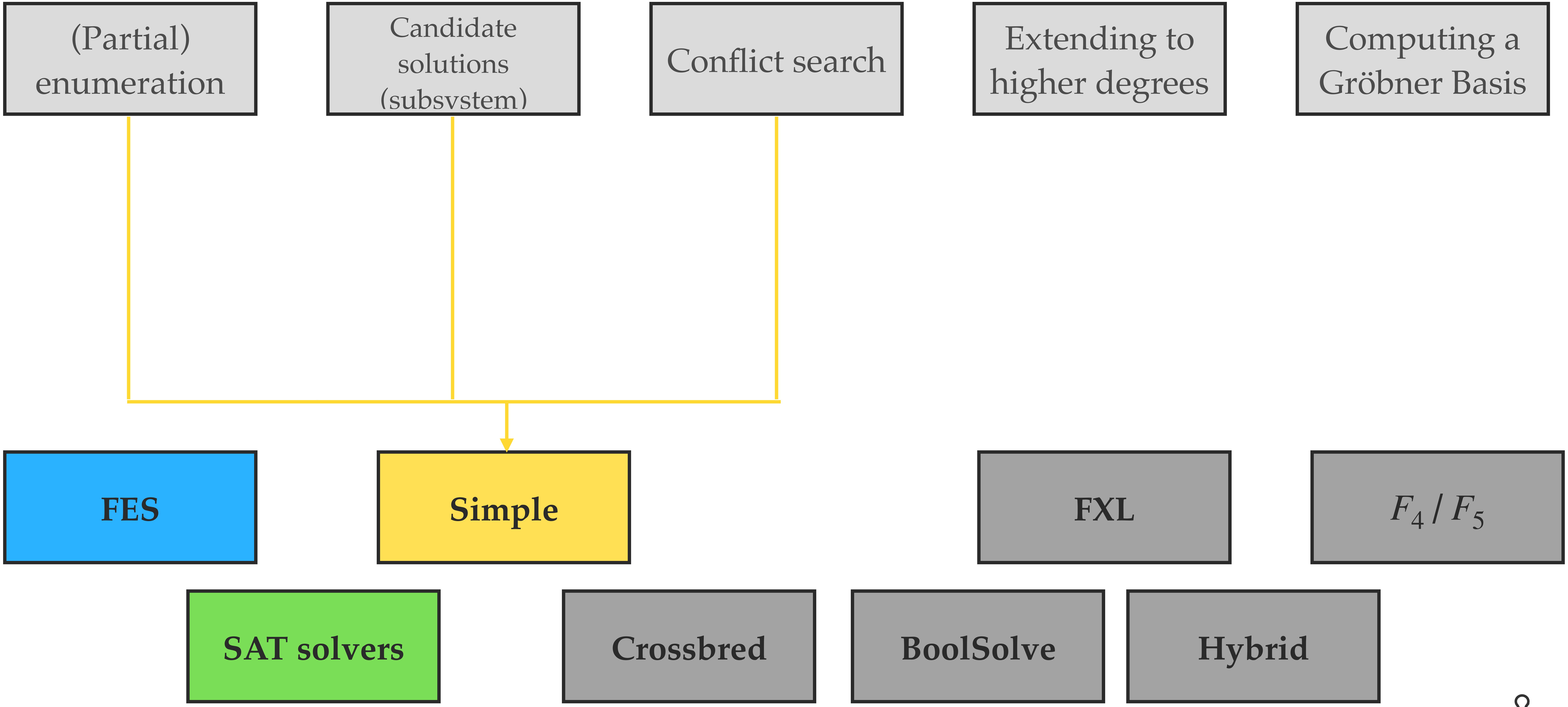
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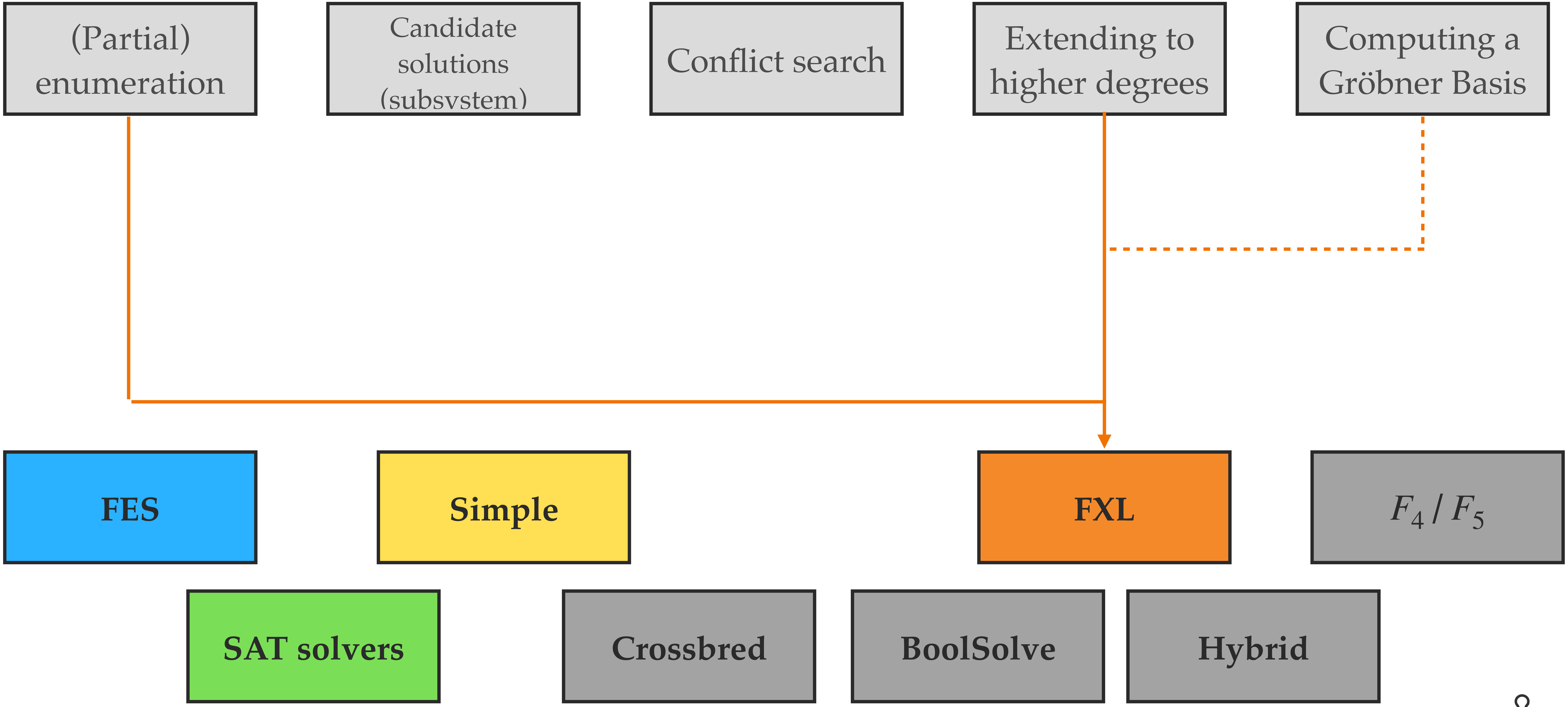
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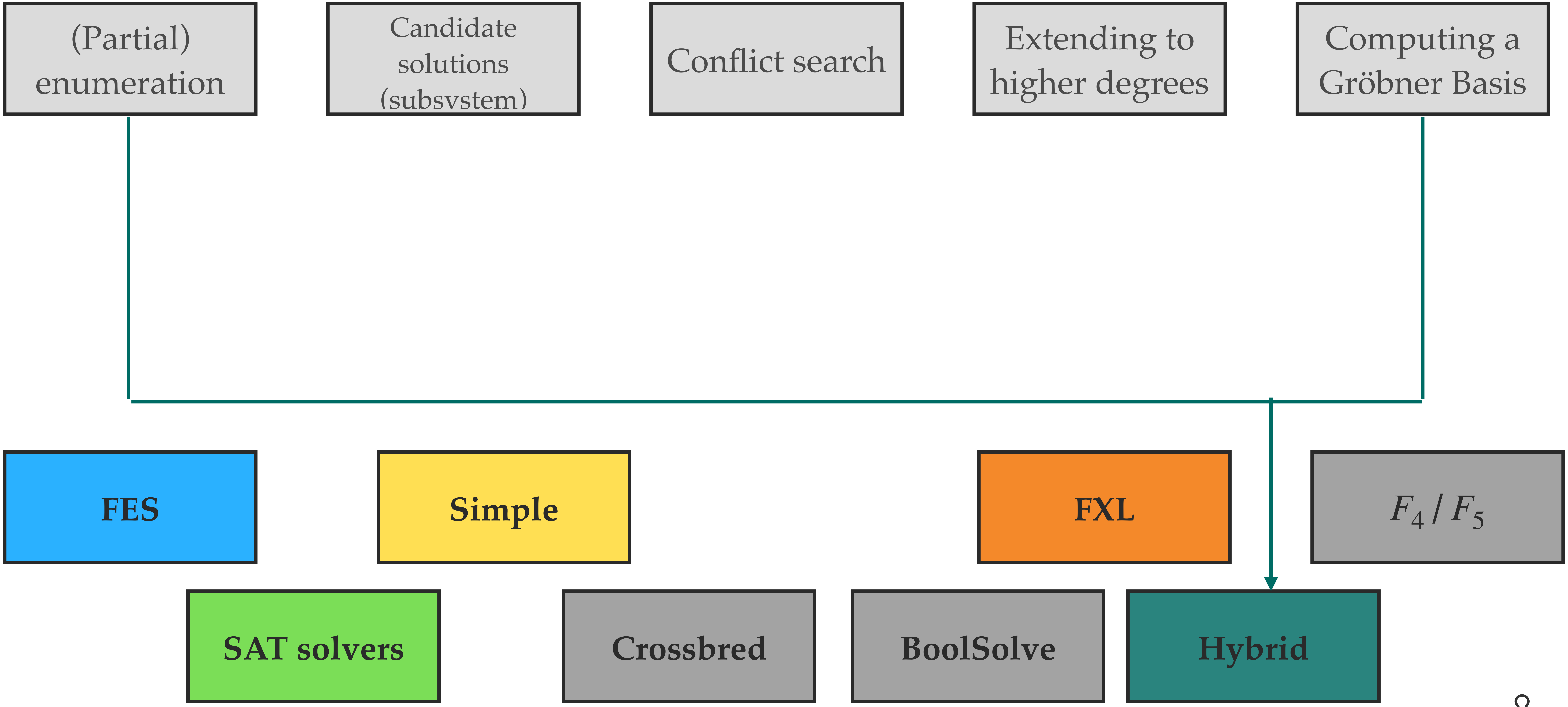
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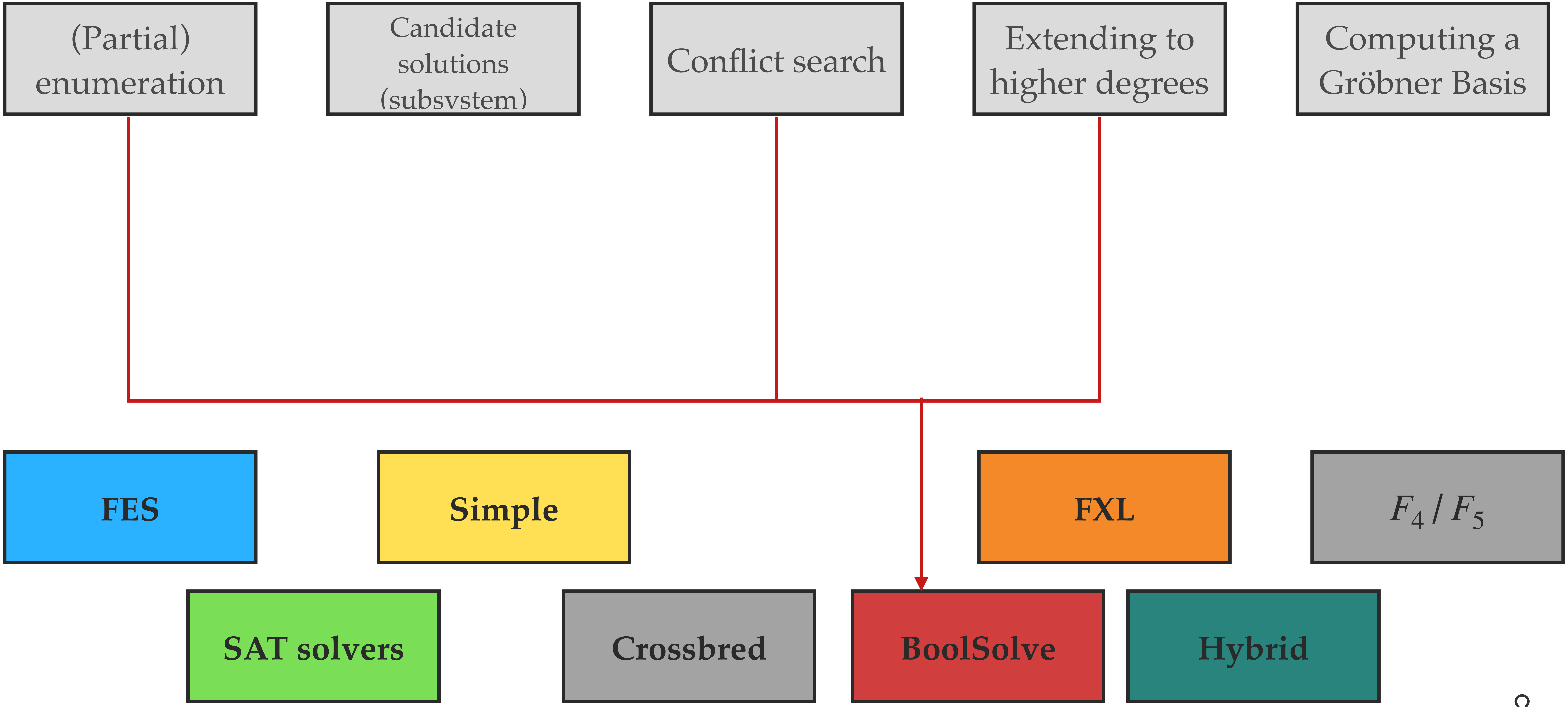

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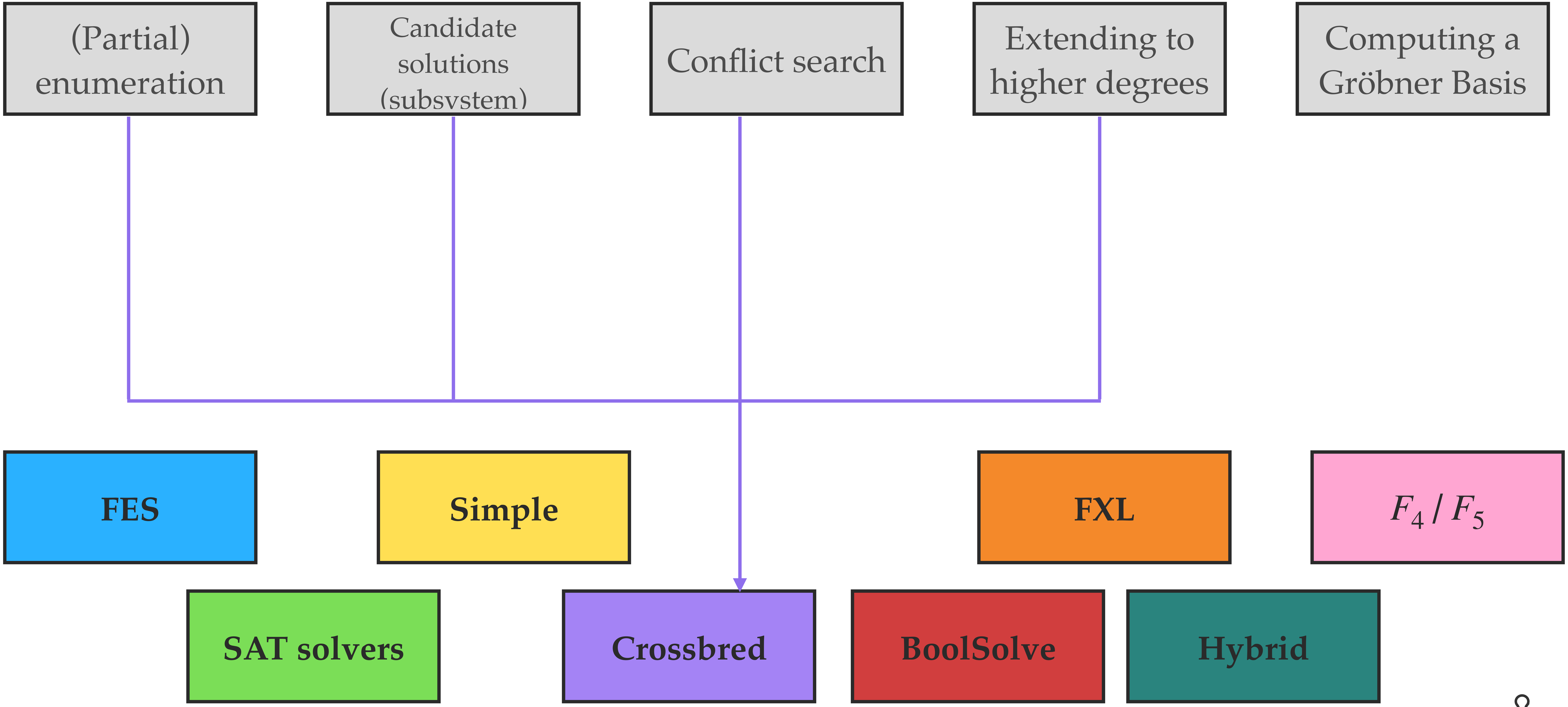
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